Surface Fitting

Moving Least Squares

Preliminaries (Algebra & Calculus)

Gradients

If F is a function assigning a real value to a 3D point, the gradient of F is the vector:

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

Preliminaries (Algebra & Calculus)

Extrema

If F is a function assigning a real value to a 3D point, then p is an extremum of F only if the gradient of F at p is zero:

$$(\nabla F)(p) = 0$$

Preliminaries (Algebra & Calculus)

Dot Products

If F has the form:

$$F(p) = \langle p, q \rangle$$

for some fixed q, then:

$$\nabla F = q$$

Preliminaries (Algebra & Calculus)

Dot Products

If F has the form:

$$F(p) = \langle p, q \rangle^k$$

for some fixed q, then:

$$\nabla F = k \langle p, q \rangle^{k-1} q$$

Preliminaries (Algebra & Calculus)

Dot Products

If F has the form:

$$F(p) = \langle p, p \rangle$$

then:

$$\nabla F = 2p$$

Preliminaries (Algebra & Calculus)

Dot Products

If F has the form:

$$F(p) = \langle p, Mp \rangle$$

then:

$$\nabla F = Mp + M^t p$$

Preliminaries (Algebra & Calculus)

Lagrangians

The extrema of F, subject to the constraint $G(p)=c$, can be found by solving:

$$(\nabla F)(p) = \lambda (\nabla G)(p)$$

(i.e. At p, the function F should only be changing in a direction that is perpendicular to the constraint.)

Preliminaries (Algebra & Calculus)

Lagrangians and Symmetric Matrices

If M is a symmetric matrix, p is an extremum of:

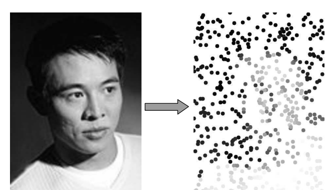
$$F(p) = p^t Mp$$

subject to the constraint $\|p\|=1$ if and only if p is an eigen-vector of M.

$$\left(\lambda p = (\nabla F)(p) = Mp + M^t p = 2Mp\right)$$

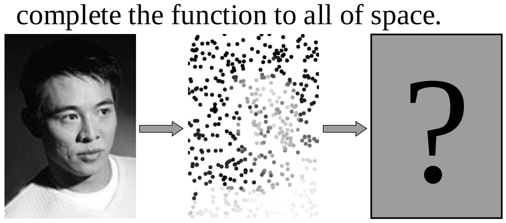
McLain (1974)

Challenge:
Given a discrete sampling of a function



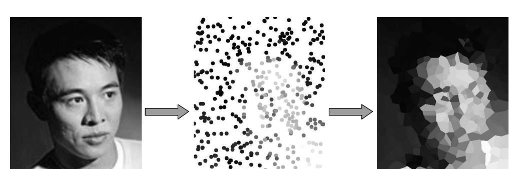
McLain (1974)

Challenge:
Given a discrete sampling of a function, complete the function to all of space.



Constant

For each point in space, we can find the closest sample point and use its value.



The reconstruction is piecewise constant

Challenge

Given a set of 2D points $\{p_i\}$, with associated real values ϕ_i, define a function Φ, defined over all of 2D space, that fits/approximates the sample data.

Weighted Averaging

Given a set of 2D points $\{p_i=(x_i,y_i)\}$, with associated real values ϕ_i, for any point $p=(x,y)$, set:

$$\Phi(p) = \frac{\sum_i \Theta(\|p_i - p\|)\ \phi_i}{\sum_i \Theta(\|p_i - p\|)}$$

Properties of the weight function $\Theta(r)$:
- It should drop off with distance
- Drop off too slow → blurring
- Drop off too sharp → numerical instability

Key Idea (McLain 1974)

This type of fitting can be viewed as function optimization:

Generalized Framework

This type of fitting can be viewed as function optimization:

Let $\mathcal{L}$ be a space of functions, at each point p, find $\phi_p \in \mathcal{L}$, minimizing the weighted error:

$$E(\phi) = \sum_i \Theta(\|p_i - p\|)|\phi_p(x_i, y_i) - \phi_i|^2$$

and set:

$$\Phi(p) = \phi_p(p)$$

Generalized Framework

In the case that $\mathcal{L}$ is the space of constant order polynomials, this reduces to solving for the real value ϕ_p that minimizes:

$$E(\phi) = \sum_i \Theta(\|p - p_i\|)|\phi_p - \phi_i|^2$$

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Thus:

$$\phi_p = \frac{\sum_i \Theta(\|p - p_i\|)\ \phi_i}{\sum_i \Theta(\|p - p_i\|)}$$

Quadratic Framework

Example:

We can let $\mathcal{L}$ be the six-dimensional space of second order polynomials:

$$\mathcal{L}=\{c_{00}+c_{10}x+c_{01}y+c_{11}xy+c_{20}x^2+c_{02}y^2\}$$

Quadratic Framework

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Then we would like to solve for the polynomial $\phi_p(x,y)$ minimizing:

$$E(\phi_p)=\sum_i \Theta(\|p-p_i\|)\left|\phi_p(x_i,y_i)-\phi_i\right|^2$$

Quadratic Framework

Solving for ϕ_p:

Let C_ϕ be the vector of coefficients:

$$C_\phi=(c_{00},c_{10},c_{01},c_{11},c_{20},c_{02})$$

And let V_i be vector:

$$V_i=(1,x_i,y_i,x_iy_i,x_i^2,y_i^2)$$

This lets us write:

$$\phi_p(x_i,y_i)=\langle C_p,V_i\rangle$$

Quadratic Framework

Solving for ϕ_p:

To solve for ϕ_p, we need to find the vector C_p, minimizing:

$$E(\phi_p)=\sum_i \Theta(\|p-p_i\|)\left|\langle C_\phi,V_i\rangle-\phi_i\right|^2$$

$$=\sum_i \Theta(\|p-p_i\|)\left[\langle C_\phi,V_i\rangle^2+\phi_i^2-2\langle C_\phi,V_i\rangle\phi_i\right]$$

Quadratic Framework

Solving for ϕ_p:

$$E(\phi_p)=\sum_i \Theta(\|p-p_i\|)\left[\langle C_\phi,V_i\rangle^2+\phi_i^2-2\langle C_\phi,V_i\rangle\phi_i\right]$$

Setting the derivative with respect to C_p equal to zero gives:

$$\sum_i \Theta(\|p-p_i\|)V_i\langle C_\phi,V_i\rangle=\sum_i \Theta(\|p-p_i\|)V_i\phi_i$$

Quadratic Framework

Solving for ϕ_p:

$$\sum_i \Theta(\|p-p_i\|)V_i\langle C_\phi,V_i\rangle=\sum_i \Theta(\|p-p_i\|)V_i\phi_i$$

Using the fact that $w\langle u,v\rangle=wu^tv$ gives:

$$\left[\sum_i \Theta(\|p-p_i\|)(V_iV_i^t)\right]C_\phi=\sum_i \Theta(\|p-p_i\|)V_i\phi_i$$

Quadratic Framework

Solving for ϕ_p:

$$\sum_i \Theta(\|p-p_i\|) V_i \langle C_\phi, V_i \rangle = \sum_i \Theta(\|p-p_i\|) V_i \phi_i$$

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$$C_\phi = \left[\sum_i \Theta(\|p-p_i\|)(V_i V_i^t)\right]^{-1} \left(\sum_i \Theta(\|p-p_i\|) V_i \phi_i\right)$$

Quadratic Framework

Solving for ϕ_p:

$$C_\phi = \left[\sum_i \Theta(\|p-p_i\|)(V_i V_i^t)\right]^{-1} \left(\sum_i \Theta(\|p-p_i\|) V_i \phi_i\right)$$

Given the optimal coefficients of the polynomial, we can now set the value at the point p:

$$\Phi(p) = \phi_p(p)$$

Using higher order functions gives us more freedom to better fit the data

MLS vs. Splines

MLS	Splines
A continuous set of functions glued together with the weight function.	A discrete set of functions designed to glue together at the end points.
Continuity defined by the continuity of the weight function.	Continuity defined by the order of the polynomial at the joints.
No structure on the distribution of the sample points.	Sample points are distributed with a regular grid topology.

What is a manifold?

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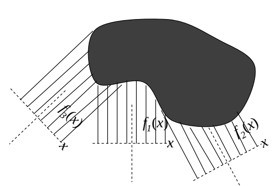
What is a manifold?

A smooth 2-manifold (embedded in 3D) is a surface, smoothly stitched together from the graphs of functions.



MLS Approach (Levin)

As in McLain, define a continuous set of graphs, and stitch them together using the weighting function:

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Given a set of points $\{r_i\}$ and a weighting function Θ, for any point r find, a graph that locally fits the data points.

MLS Approach (Levin)

As in McLain, define a continuous set of graphs, and stitch them together using the weighting function:

1. For each point, define locally “best fitting” plane

2. Using the height of the points from the plane as the sample value, apply MLS to complete the function.

MLS Approach (Basic Approach)

Given a set of points $\{r_i\}$ and a weighting function Θ, for any point r find, a graph that locally fits the data points.

1. <u>Plane fitting</u>: Solve for the plane with normal a_r and containing the point q_r, that minimizes:

$$\begin{aligned} E(a_r, q_r) &= \sum_i \Theta(\|r - r_i\|)\langle a_r, r_i - q_r\rangle^2 \\ &= \sum_i \Theta(\|r - r_i\|)\left(\langle a_r, r_i\rangle^2 + \langle a_r, q_r\rangle^2 - 2\langle a_r, r_i\rangle\langle a_r, q_r\rangle\right) \end{aligned}$$

Plane Fitting (Basic Approach)

$$E(a_r, q_r) = \sum_i \Theta(\|r - r_i\|)\left(\langle a_r, r_i\rangle^2 + \langle a_r, q_r\rangle^2 - 2\langle a_r, r_i\rangle\langle a_r, q_r\rangle\right)$$

<u>Solving for q_r</u>

Taking the derivative with respect to q_r and setting the result equal to zero gives:

$$\sum_i \Theta(\|r - r_i\|)\langle a_r, q_r\rangle = \sum_i \Theta(\|r - r_i\|)\langle a_r, r_i\rangle$$

Plane Fitting (Basic Approach)

$$E(a_r, q_r) = \sum_i \Theta(\|r - r_i\|)\left(\langle a_r, r_i\rangle^2 + \langle a_r, q_r\rangle^2 - 2\langle a_r, r_i\rangle\langle a_r, q_r\rangle\right)$$

<u>Solving for q</u>

Taking the derivative with respect to q and setting the result equal to zero gives:

$$\sum_i \Theta(\|r - r_i\|)\langle a_r, q_r\rangle = \sum_i \Theta(\|r - r_i\|)\langle a_r, r_i\rangle$$

$$\Updownarrow$$

$$q_r = \frac{\sum_i \Theta(\|r - r_i\|) r_i}{\sum_i \Theta(\|r - r_i\|)} + a_r^\perp$$

Plane Fitting (Basic Approach)

$$E(a_r, q_r) = \sum_i \Theta(\|r - r_i\|)\left(\langle a_r, r_i\rangle^2 + \langle a_r, q_r\rangle^2 - 2\langle a_r, r_i\rangle\langle a_r, q_r\rangle\right)$$

<u>Solving for a_r</u>

Using the fact that $w\langle u,v\rangle = wu^t v$ gives:

$$E(a_r, q_r) = a_r^t\left[\sum_i \Theta(\|r - r_i\|)\left(r_i r_i^t + q_r q_r^t - r_i q_r^t - q_r r_i^t\right)\right] a_r$$

Plane Fitting (Basic Approach)

$$E(a_r,q_r) = a_r^t\left[\sum_i \Theta(\|r-r_i\|)(r_i r_i^t + q_r q_r^t - r_i q_r^t - q_r r_i^t)\right]a_r$$

Solving for a_r

Since the matrix:

$$\sum_i \Theta(\|r-r_i\|)(r_i r_i^t + q_r q_r^t - r_i q_r^t - q_r r_i^t)$$

is symmetric, a_r must be an eigen-vector.

MLS Approach (Basic Approach)

2. Function fitting: Fix an orthonormal basis $\{v_r,w_r\}$ perpendicular to a_r, and express r_i as the 2D sample:

$$\tilde{r}_i = (\langle r_i - q_r, v_r\rangle, \langle r_i - q_r, w_r\rangle)$$

with value:

$$\phi_i = \langle r_i - q_r, a_r\rangle$$

Now use the McLain approach to fit a quadratic polynomial $P_r(x,y)$ as the graph.

MLS Approach (Basic Approach)

Projection:

Given a set of points $\{r_i\}$ and a weighting function Θ, for any point r:

1. Find the fitting plane (a_r,q_r) at r,
2. Find the fitting polynomial $P_r(x,y)$ at r,
3. Express r as: $r = q_r + x_r v_r + y_r w_r + z_r a_r$ $\begin{pmatrix} x_r = \langle r-q_r, v_r\rangle \\ y_r = \langle r-q_r, w_r\rangle \\ z_r = \langle r-q_r, a_r\rangle \end{pmatrix}$

Set the "projection" of r to be:

$$\boxed{\pi(r) = q_r + x_r v_r + y_r w_r + P_r(x,y)a_r}$$

MLS Approach (Basic Approach)

Advantage:

- Gives a way of sending points to the surface without explicitly defining where the surface is.

Limitations:

- The "projection" operator π is not actually a projection: $\pi(\pi(r)) \neq \pi(r)$.
- If r is close to the surface, $\pi(r)$ will not necessarily be mapped onto the approximating surface.

MLS Approach (Levin)

The "projection" operator is not actually a projection because:

1. The fitting plane computed for r is not the same fitting plane computed for $\pi(r)$
2. The graph fitted to r is not the same graph fitted to $\pi(r)$

since the weighting coefficients change:

$$\Theta(\|r_i - r\|) \neq \Theta(\|r_i - \pi(r)\|)$$

MLS Approach (Levin)

Problem

• The fitting plane computed for r is not the same fitting plane computed for $\pi(r)$

Observation

The projection $\pi(r)$ only moves r along the normal direction of the plane.

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Observation

The projection $\pi(r)$ only moves r along the normal direction of the plane.

Solution

Modify the definition of "best fitting" plane so that it (locally) only depends on the line from r in the direction of a_r.

MLS Approach (Levin)

Problem

2. The graph fitted to r is not the same graph fitted to $\pi(r)$

$$\sum_i \Theta(\|r_i - r\|)(P(\tilde{r}_i) - \phi_i) \neq \sum_i \Theta(\|r_i - \pi(r)\|)(P(\tilde{r}_i) - \phi_i)$$

MLS Approach (Levin)

Problem

2. The graph fitted to r is not the same graph fitted to $\pi(r)$

$$\sum_i \Theta(\|r_i - r\|)(P(\tilde{r}_i) - \phi_i) \neq \sum_i \Theta(\|r_i - \pi(r)\|)(P(\tilde{r}_i) - \phi_i)$$

Solution

Modify the weighting for the "best fitting" function so that it only depends on q_r and not on r:

$$\sum_i \Theta(\|r_i - r\|)(P(\tilde{r}_i) - \phi_i) \quad \rightarrow \quad \sum_i \Theta(\|r_i - q_r\|)(P(\tilde{r}_i) - \phi_i)$$

MLS Approach (Basic Approach)

Advantages:

- The "projection" operator π is a projection for points sufficiently close to the surface: $\pi(\pi(r))=\pi(r)$
- If r is close to the surface, $\pi(r)$ will be mapped onto the approximating surface.

MLS Approach (Basic Approach)

Disadvantages:

- By changing the fitting function:
  $$\sum_i \Theta(\|r_i - r\|)\langle a_r, r_i - q_r\rangle^2 \rightarrow \sum_i \Theta(\|r_i - q_r\|)\langle a_r, r_i - q_r\rangle^2$$
  it is now necessary to optimize the fitting plane over the weighting functions, which can be very difficult.