600.363/463 Algorithms
Mid Semester Examination 1
October 11, 2013
1 hr 10 mins; closed book

I. Specify whether the following equations hold or not. Give an intuitive jusfification for your answer. Formal proofs are not needed.

1. $3n^2 + 5n \;=\; O(n^2 \log n)$
2. $2^n \;=\; O(n^4 \log\log n)$
3. $10n^2 \;=\; O(n^2 - 100n)$
4. $(n^2 + n)(n \log n + n^{1.1}) \;=\; O(n^3)$
5. $n^3 \log n \;=\; \Theta(n^3)$
6. $n^3 \log^5 n \;=\; O(n^{3.01})$
7. $4^n \;=\; \Theta(3^n)$
8. $n^2 \log n (\log\log n)^4 \;=\; O(n^2 \log^2 n)$
9. $\log^2 n \;=\; O(n^{0.05})$
10. $n^2 + n^{1.9} \;=\; O(n^2 + n^{0.5})$

II. Solve the following recurrences. You can invoke a known result and write down the answer, or you can justify the answer by repeated substitutions or by induction. When needed you can assume approve initial values.

1. $T(n) \leq 3T(\frac{n}{2}) + n^2$
2. $T(n) \leq 5T(\frac{n}{2}) + n^2$
3. $T(n) \leq 2T(\frac{n}{4}) + T(\frac{n}{3})$
4. $T(n) \leq 2nT(n-1)$
5. $T(n) \leq 2T(n-1) + T(n-2)$

III. Let δ be a positive-valued *affinity* function between pairs of symbols. Then for any pair of equal length strings $X = a_1 a_2 \cdots a_\ell$ and $Y = b_1 b_2 \cdots b_\ell$, $\delta(X,Y) = \sum_{i=1}^{\ell} \delta(a_i, b_i)$. For any 2 strings $X = a_1 a_2 \cdots a_n$, $Y = b_1 b_2 \cdots b_m$, $n \leq m$, $\delta(X,Y)$ is defined as

$$\max_{|Z|=n,\ Z \text{ a subsequence of } Y} \delta(X,Z)$$

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For example, if $\delta(a,a) = 2$, $\delta(a,b) = \delta(b,a) = 1$, $\delta(b,b) = 3$, then $\delta(abab, bbab) = 1+3+2+3 = 9$; $\delta(abba, \underline{bb}a\underline{aa}) = 7$, and a subsequence that maximizes the value is underlined.

Design a dynamic programming based algorithm which for a given δ, X and Y computes $\delta(X, Y)$.

Hint: If $X = x_1 x_2 \cdots x_n$, $Y = y_1 y_2 \cdots y_m$, for any $i \leq j$, let $c[i,j] = \delta(x_1 x_2 \cdots x_i, y_1 y_2 \cdots y_j)$.

a) Specify a computation formula for $c[i,j]$,

b) Specify an order in which the entries of the c matrix can be computed, and

c) What is the speed of the algorithm?

IV.

1. Assume that a set S of n numbers are stored in some form of balanced binary search tree; i.e. the depth of the tree is $O(\log n)$. In addition to the key value and the pointers to children, assume that every node contains the number of nodes in its subtree. Design $O(\log n)$ step algorithms for performing the following operations.

   (a) Given a positive integer k, $1 \leq k \leq n$, compute the k^{th} smallest element of S, and

   (b) Given 2 numbers x and y, compute the size of the subset $\{z| \ z \in S, \ x \leq z \leq y\}$.

2. Specify a reason why a balanced binary tree is better than a complete binary tree for storing the set S:

   (a) A balanced binary tree is more space efficient,

   (b) Inserts and Deletes can be performed faster, or

   (c) Implementation of a balanced binary tree requires less RAM space