Lecture 4 and 5

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1 Least squares example

from Paul's Linear Algebra Notes

Solving over-defined least-squares problem with normal system of equations.

$$\begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$x = (A^T A)^{-1} A^T b$$

Note, $(A^T A)^{-1} A^T$ is the pseudo-inverse of A.

$$\begin{bmatrix} -3 & 1 & -7 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 & -7 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix}$$

$$\begin{bmatrix} 84 & -4 \\ -4 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1134 \\ 166 \end{bmatrix}$$

$$x_1 = -12.1, x_2 = 29.4$$

2 Least squares using SVD

$$A_{p \times k} x = b, p > k.$$

Solve for x.

$$USV^T = svd(A)$$

$$U = p \times p$$

columns are left singular vectors of A (full rank, unitary)

$$S = p \times k$$

diagonal matrix of the singular values of A.

$$V = k \times k$$

columns are right singular vectors of A (full rank, unitary)

- U and V^T can be inverted using the conjugate transpose
- S can be inverted using one-over each diagonal element.

Solve for x using

$$USV^T x = b$$

$$x = VS^{-1}U^T b$$

3 Nullspace problem

$$\arg\min_x \|Ax\|$$

Set

$$Ax = \mathbf{0}$$

and solve for non-trivial x.

from Paul's Linear Algebra Notes

3.1 example 1

$$\begin{bmatrix} 2 & 0 \\ -4 & 10 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 = 0, -4x_1 + 10x_2 = 0$$

$$\Rightarrow x_1 = 0, x_2 = 0$$

3.2 example 2

$$\begin{bmatrix} 1 & -7 \\ -3 & 21 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - 7x_2 = 0, -3x_1 + 21x_2 = 0$$

$$\Rightarrow x_2 = t, x_1 = 7t$$

What is the rank of the matrix in these two examples and what is their effect on data?

- In example 1, the only solution is the trivial **0** solution, so full rank.
- In example 2, there is a class of solutions that satisfy.

(Matlab demonstration: Project some 2D data points using the matrices to demonstrate the effect of the non-full rank projection.)

4 Homography

$$H_{3\times3} \cdot P_1 \propto P_2$$

$$P_1 = \{x_1, y_1, w_1\}^T$$
$$P_2 = \{x_2, y_2, w_2\}^T$$

$$\widetilde{P_2} = \begin{bmatrix} 0 & -w_2 & y_2 \\ w_2 & 0 & -x_2 \\ -y_2 & x_2 & 0 \end{bmatrix}$$

$$P_2 \times P_1 = \widetilde{P_2} \cdot P_1.$$

- What is the resulting vector from the cross product of $P_2 \times P_1$?
- It is orthogonal to the plane generated by P_1 and P_2.
- It has a magnitude proportional to the angle between P_1 and P_2.
- If P_1 and P_2 are *parallel*, then $P_2 \times P_1 = 0$.
- If $P_2 \times P_1 = 0$, then P_1 and P_2 are proportional.

4.1 Solving for H

$$H_{3\times3} \cdot P_1 \propto P_2 \Rightarrow \widetilde{P_2} H P_1 = 0$$

With some re-arranging

$$\widetilde{P_2} \begin{bmatrix} x_1 & y_1 & w_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & w_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1 & w_1 \end{bmatrix} \cdot \begin{bmatrix} h_{1,1} \\ h_{2,1} \\ \vdots \\ h_{3,3} \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & 0 & -w_2x_1 & -w_2y_1 & -w_2w_1 & y_2x_1 & y_2y_1 & y_2w_1 \\ -w_2x_1 & -w_2y_1 & -w_2w_1 & 0 & 0 & 0 & -x_2x_1 & -x_2y_1 & -x_2w_1 \\ -y_2x_1 & -y_2y_1 & -y_2w_1 & -x_2x_1 & -x_2y_1 & -x_2w_1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} h_{1,1} \\ h_{2,1} \\ \vdots \\ h_{3,3} \end{bmatrix} = \mathbf{0}$$

Now we have a Null-space formulation. We are only solving for H "up-to-scale" so any non-trivial solution is useful.