Non-reversible jump algorithms for Bayesian nested model selection

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1. Running example

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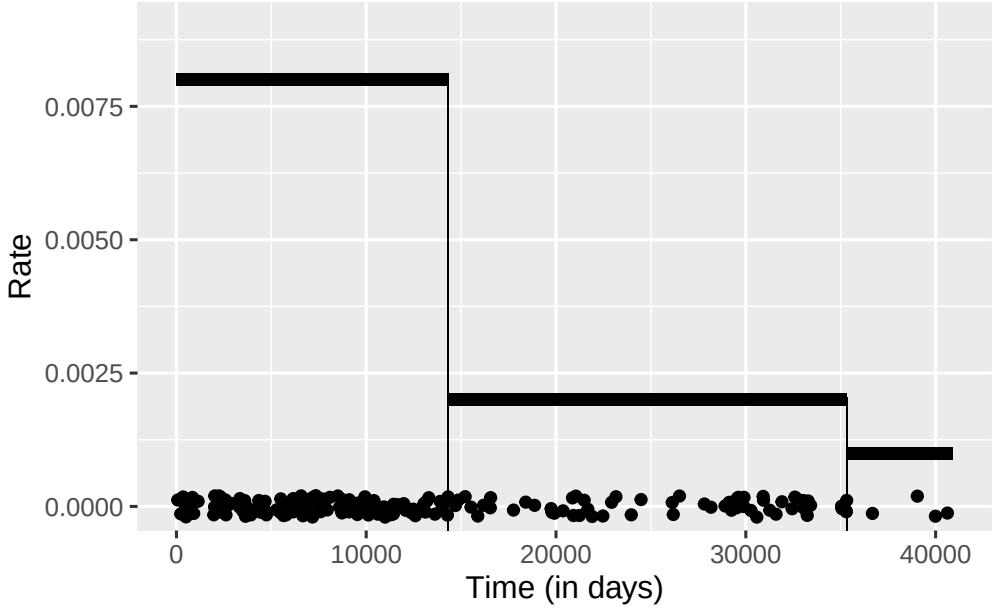

MH/fixed-dimensional MCMC sampler

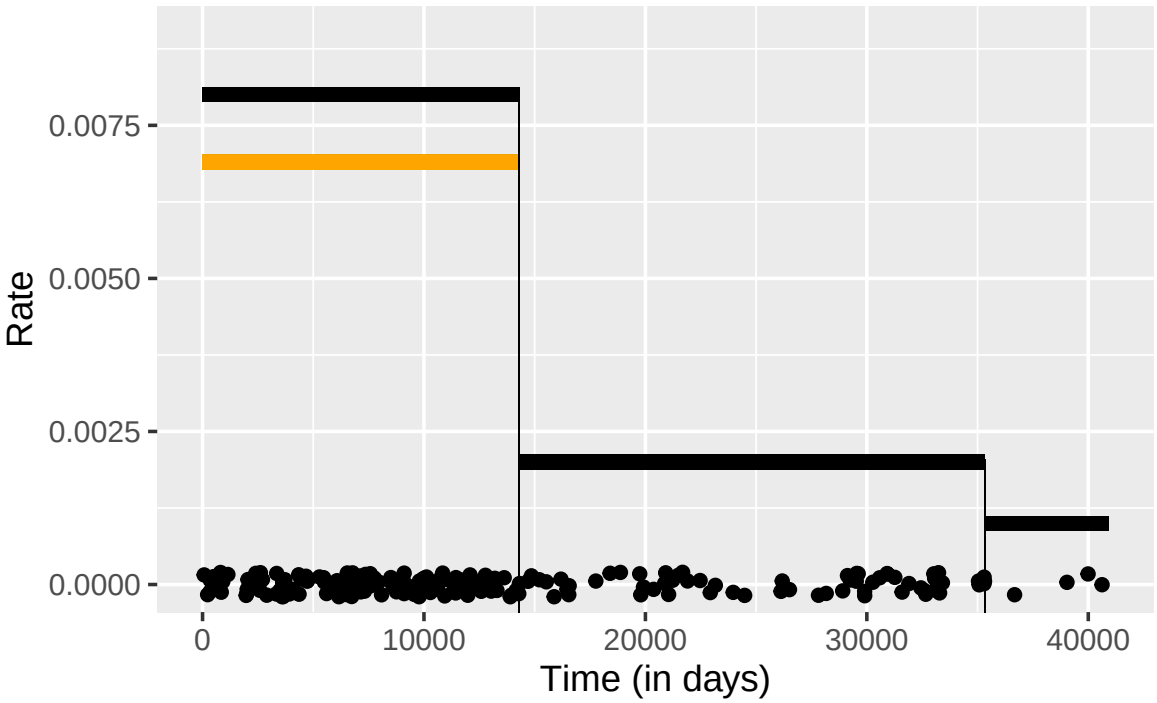

Figure 2: Proposal $\mathbf{y}$ with $\alpha(\mathbf{x}, \mathbf{y}) = 0.08$

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  - d_k: number of parameters in Model k.

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3. Transform $(\mathbf{x}_k, \mathbf{u}_{k \mapsto k'})$: $T_{k \mapsto k'}(\mathbf{x}_k, \mathbf{u}_{k \mapsto k'}) := (\mathbf{y}_{k'}, \mathbf{u}_{k' \mapsto k})$. If

$$u_a \sim \mathcal{U}(0,1) \leq \alpha_{\mathrm{RJ}}((k, \mathbf{x}_k), (k', \mathbf{y}_{k'}))$$
$$:= 1 \wedge \frac{\pi(k', \mathbf{y}_{k'})\, g(k', k)\, q_{k' \mapsto k}(\mathbf{u}_{k' \mapsto k})}{\pi(k, \mathbf{x}_k)\, g(k, k')\, q_{k \mapsto k'}(\mathbf{u}_{k \mapsto k'})\, |J_{T_{k \mapsto k'}}(\mathbf{x}_k, \mathbf{u}_{k \mapsto k'})|^{-1}},$$

   set the next state of the chain to $(k', \mathbf{y}_{k'})$. Otherwise, set it to $(k, \mathbf{x}_k)$.
4. Go to Step 1.

Example (cont'd)

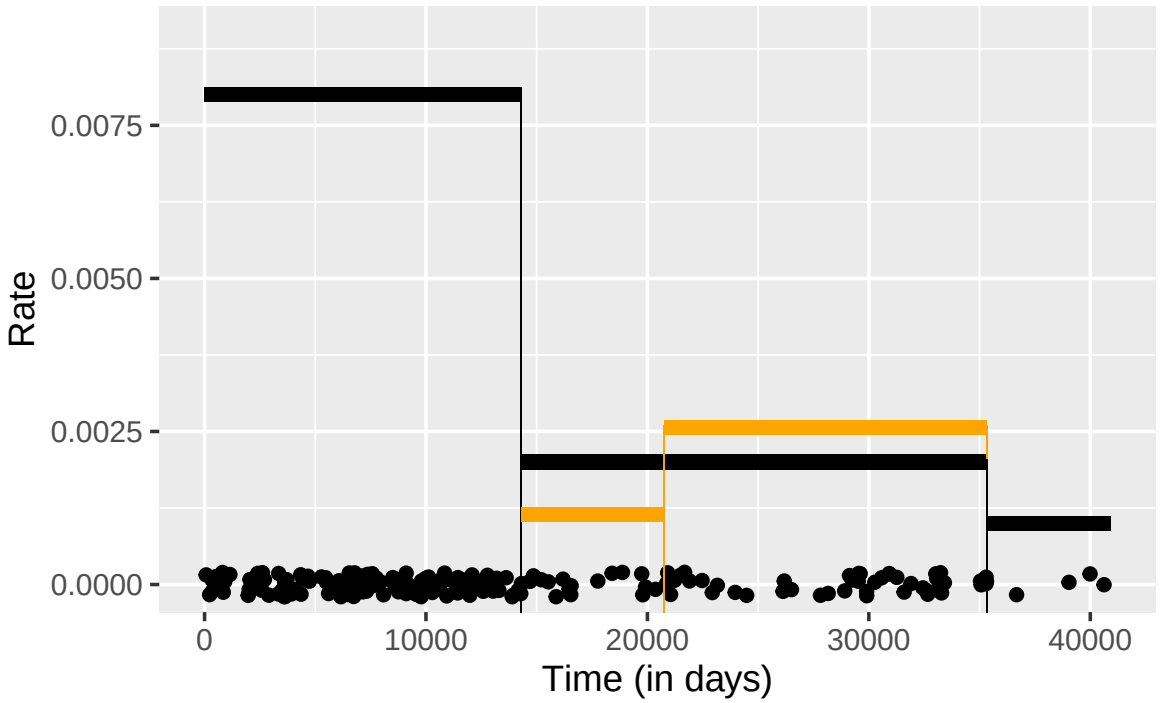

Figure 3: Proposal $(k', \mathbf{y}_{k'})$ with $\alpha_{\mathrm{RJ}}((k, \mathbf{x}_k), (k', \mathbf{y}_{k'})) = 1$

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RJ Algorithms (Cont'd)

- Usually,

$$g(k,k') = \begin{cases} \tau & \text{if} \quad k' = k, \\ (1-\tau)/2 & \text{if} \quad k' = k-1 \text{ or } k+1. \end{cases}$$

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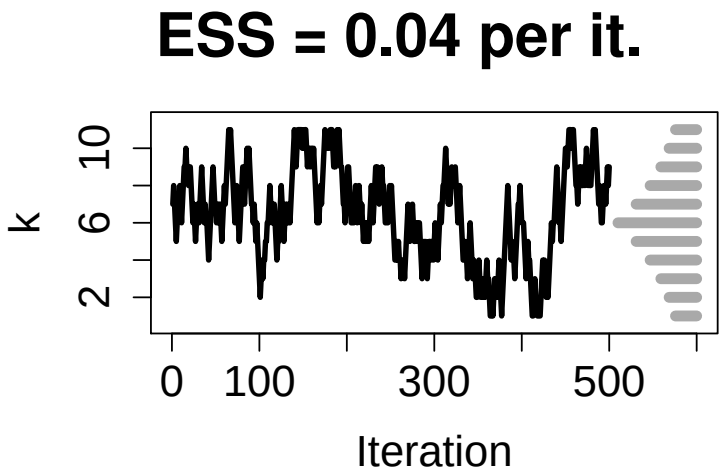
- **Remark**: the model space $\mathcal{K}$ is explored through a random walk.



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NRJ VS RJ

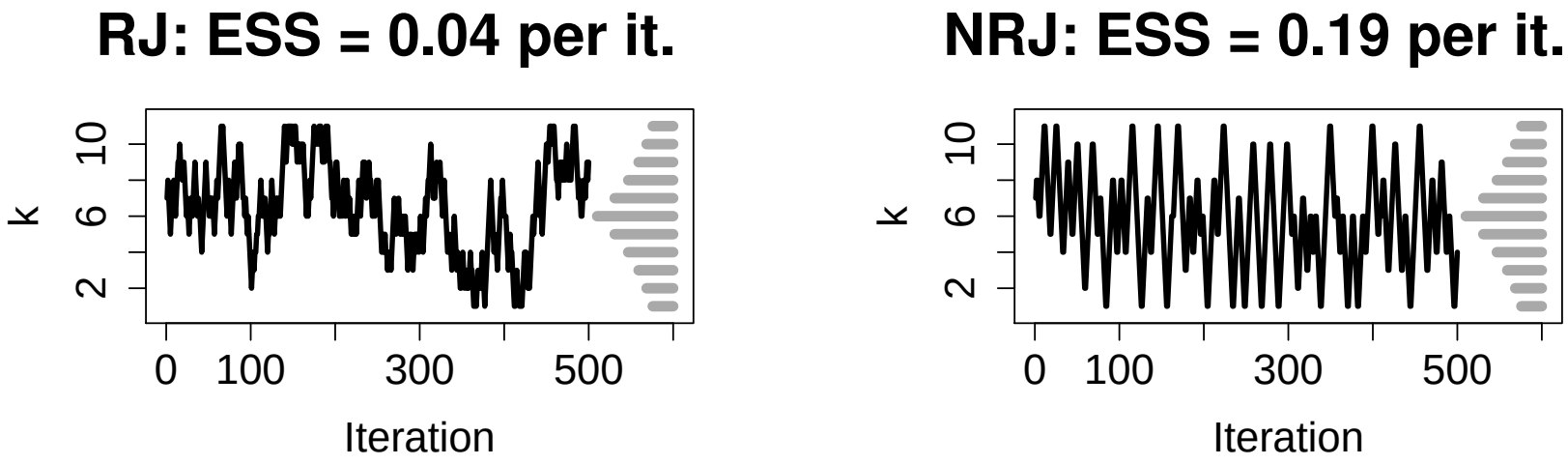

Figure 5: Trace plots for *ideal* RJ selecting uniformly at random which model to switch to and NRJ algorithms, and showing only the iterations in which model switches are proposed

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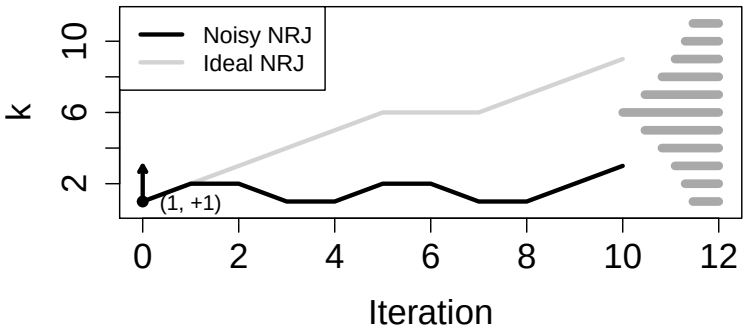
- Typically:

$$\alpha_{\mathrm{NRJ}}((k, \mathbf{x}_k), (k', \mathbf{y}_{k'})) = 1 \wedge \frac{\pi(k')}{\pi(k)} \varepsilon(\mathbf{x}_k, \mathbf{y}_{k'}).$$



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- **Fact:** Markov chains simulated by NRJ converge weakly to those produced by ideal NRJ.

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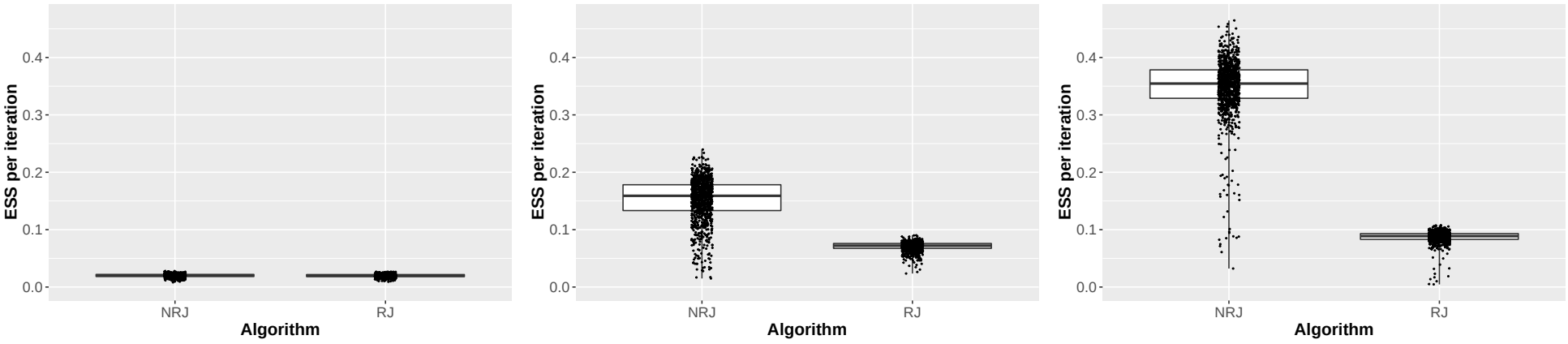

(a) Vanilla samplers **(b) In between** **(c) Ideal samplers**

Figure 8: ESS for NRJ and RJ when the samplers are: (a) vanilla samplers, (b) in between samplers, (c) ideal samplers

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- Questions?

Andrieu, C., Doucet, A., Yıldırım, S., and Chopin, N. (2018).

On the utility of Metropolis–Hastings with asymmetric acceptance ratio.
arXiv:1803.09527.

Karagiannis, G. and Andrieu, C. (2013).
Annealed importance sampling reversible jump MCMC algorithms.
J. Comp. Graph. Stat., 22(3):623–648.