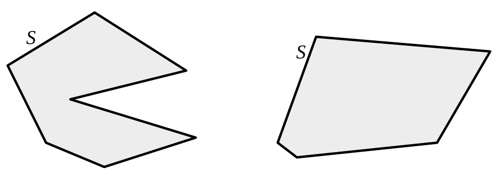
Computational Geometry: Convex Hulls

Outline

- Definitions
- Algorithms

Definition I

A set S is <u>convex</u> if for any two points $p,q \in S$, the line segment $pq \subset S$.



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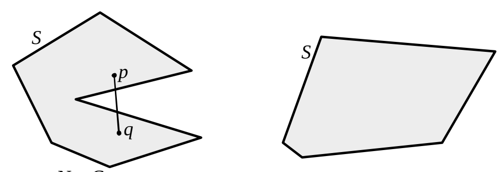

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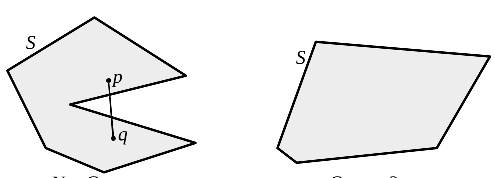

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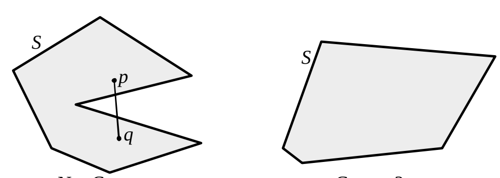
Convex?

Definition II

A set S is <u>convex</u> if it is the intersection of (possibly infinitely many) half-spaces.



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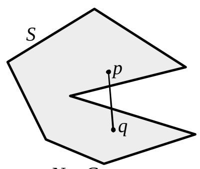

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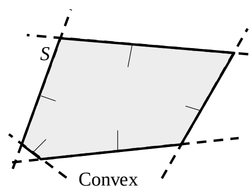

Convex

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Convex Hull

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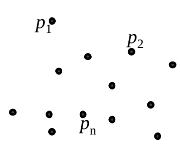
Given a finite set of points $P=\{p_1,\ldots,p_n\}$, the convex hull of P is the smallest convex set C such that $P\subset C$.



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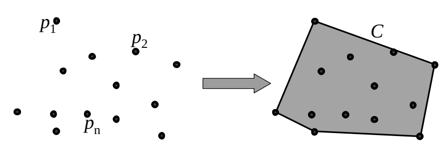
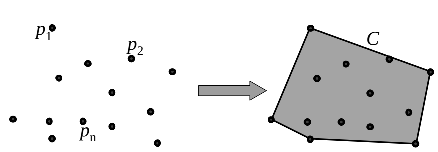


Examples

Two Dimensions:

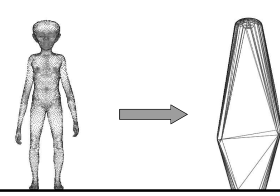
The convex hull of $P=\{p_1,\ldots,p_n\}$ is a set of line segments with endpoints in P.



Examples

Three Dimensions:

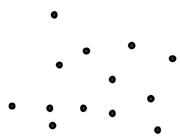
The convex hull of $P=\{p_1,\ldots,p_n\}$ is a triangle mesh with vertices in P.



Algorithms

Brute Force (2D):

Given a set of points *P*, test each line segment to see if it makes up an edge of the convex hull.



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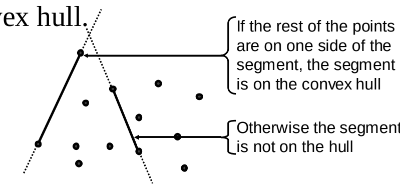

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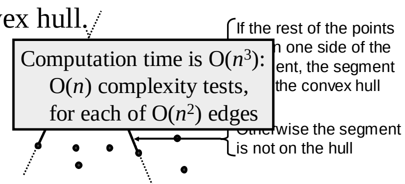

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In a *d*-dimensional space, the complexity is O(n^{d+1}).

Algorithms

There are many algorithms for computing the convex hull:

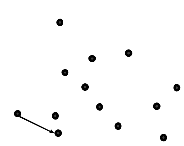
- Brute Force: O(n^3)
- Gift Wrapping
- Quickhull
- Divide and Conquer

Gift Wrapping

Key Idea:

Iteratively growing edges of the convex hull, we want to turn as little as possible.

- Given a directed edge on the hull…



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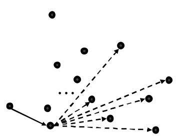

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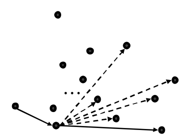

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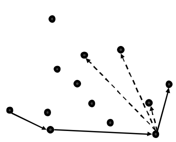

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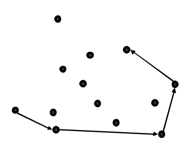

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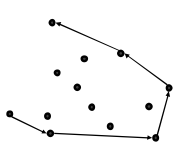

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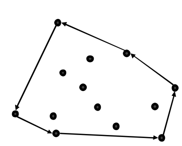

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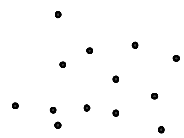
For all $a,b,c \in P$, the points contained in $\Delta abc \cap P$ cannot be on the convex hull.



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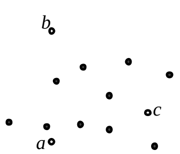

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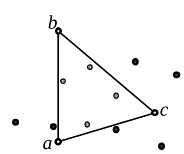

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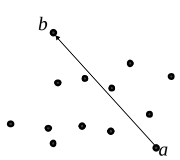

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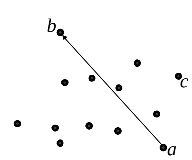

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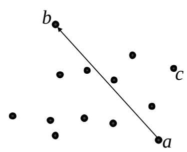

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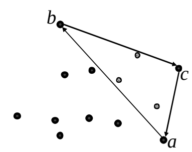

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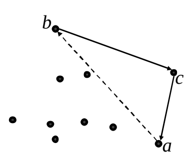

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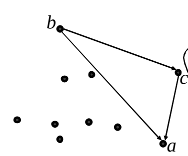

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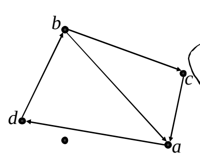

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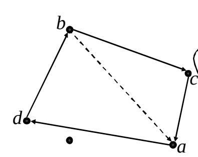

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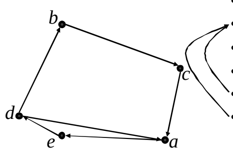

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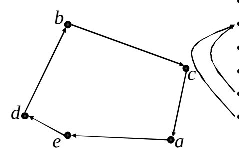

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Algorithms

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- Divide and Conquer

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Key Idea:

Finding the convex hull of small sets is easier than finding the hull of large ones.

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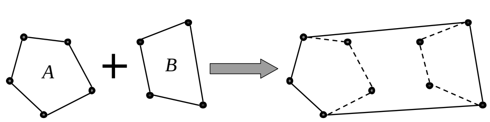
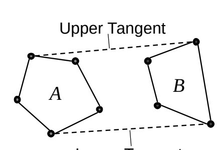
All we need is a fast way to merge hulls.



Divide and Conquer

Merging Hulls:

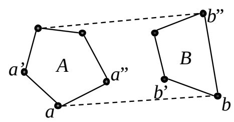
Need to find the tangents joining the hulls.



Divide and Conquer

Observation:

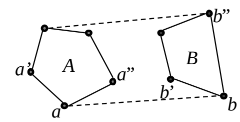
The edge $\overline{ab}$ is a tangent if the two points about *a* and the two points about *b* are on the same side of $\overline{ab}$.



Divide and Conquer

Proof:

The edge $\overline{ab}$ is a tangent if the points on both hulls are all on one side of it.

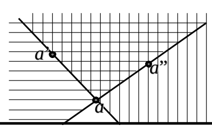


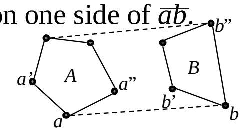
Divide and Conquer

Proof:

The edge $\overline{ab}$ is a tangent if the points on both hulls are all on one side of it.

If *a*’ and *a*” are on the same side of $\overline{ab}$, then all of *A* must be on one side of $\overline{ab}$.

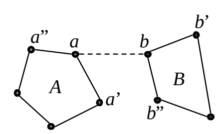




Divide and Conquer

Tangent Algorithm:

– Find an edge $\overline{ab}$ between *A* and *B* that does not intersect the two hulls.



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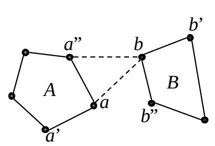
Tangent Algorithm:

– Find an edge $\overline{ab}$ between *A* and *B* that does not intersect the two hulls.

– While *a*’ and *a*” are not to the left of $\overline{ab}$, rotate *a* clock-wise.



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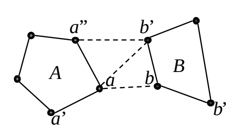

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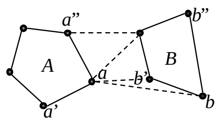

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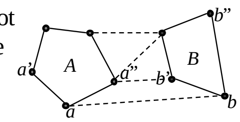

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