600.464/664 Randomized Algorithms
Final Examination
May 11, 2008
In-class, Closed Book
Time: 2 hrs 30 mins.

I. Random variables $A$, $B$, and $C$ are -1, 0, +1 valued. They are independent and uniformly distributed. Given that $P[A + B + C \leq -2] + P[A + C \leq 0] < 1$, fix a value for the r.v. $A$ by the conditional probability method of derandomization.

II. Explain the 2 stage hashing scheme that achieves $O(1)$ step response time to Member operations.

III. Extend the normal n-dimensional hypercube to $3^n$ size n-dimensional directed hypercube $\{0,1,2\}^n$ as follows. Each vertex is addressed by $\{i_1, i_2, \ldots, i_n\}$ in which each $i_j \in \{0,1,2\}$, and this vertex has edges to $n$ neighbors, the $j^{th}$ neighbor being $\{i_1, \ldots, i_{j-1}, i_j', i_{j+1}, \ldots, i_n\}$ in which $i_j' = i_j + 1 (\text{mod } 3)$. Consider a random walk on this graph in which at any vertex, the probability of walking to each of its $n$ neighbors is $1/2n$, and probability of staying at the vertex is $1/2$. Apply the coupling method and derive a good upperbound for $\tau(\epsilon)$.

IV. Let $U = \{0,1, \ldots , m-1\}$ where $m > n^3$. A class $C$ of functions from $U$ to $R = \{0,1, \ldots , n^3-1\}$ is $cfgood$ if for every size $n$ subset $S$ of $U$ there exists a function $f$ in $C$ which maps elements of $S$ into $R$ without any collisions; i.e. $(\forall S \subseteq U, |S| = n)(\exists f \in C)(\forall x, y \in S, x \neq y)(f(x) \neq f(y))$. By applying the probabilistic method, find a suitable upper bound for the size of a $cfgood$ class $C$. (Hint: Choose $k$ independent random functions, and derive an inequality for insuring the probability of failure to be less than 1.)