I. \[ \delta(i) = \sum_{j=1}^{n^2} X_{ij} \text{ in which } X_{ij} = 1 \text{ if } j^{th} \text{ ball falls into bin } i \]

\[ E(X_{ij}) = \frac{1}{n} \]

\[ \text{hence } E(\delta(i)) = \frac{n^2}{n} = n \]

Similarly \( E(\delta(j)) = n \).

Failure happens if \( \exists i,j \text{ s.t. } \delta(i) - \delta(j) > c\sqrt{n}\ln n \).

Fix \( i,j \):

\[ P(\delta(i) - \delta(j) > c\sqrt{n}\ln n) = P(\delta(i) > n + \frac{c}{2}\sqrt{n}\ln n) + P(\delta(j) < n - \frac{c}{2}\sqrt{n}\ln n) \]

[There is nothing special about the offset; it could be any value. But when it is \( n \), we can apply the Chernoff bounds.]

\[ P(\delta(i) > n + \frac{c}{2}\sqrt{n}\ln n) \leq e^{-n \frac{c^2}{8} \ln \frac{1}{2}} = \frac{1}{n^{3/8}} \]

\[ P(\delta(j) < n - \frac{c}{2}\sqrt{n}\ln n) \leq e^{-n \frac{c^2}{8} \ln \frac{1}{2}} = \frac{1}{n^{3/8}} \]

Make \( \frac{c^2}{8} \geq 3 \); i.e. \( c^2 \geq 24 \).

\( c = \frac{6}{\sqrt{3}} \) is fine.

Then \( P(\delta(i) - \delta(j) > c\sqrt{n}\ln n) \leq \frac{1}{n^{3/8}} + \frac{1}{n^{3/8}} = \frac{2}{n^{3/8}} \).

There are \( n(n-1) \) ways of choosing \((i,j)\)’s.

Hence \( P(\text{there exists } i,j \text{ s.t. } \delta(i) - \delta(j) > c\sqrt{n}\ln n) \leq \frac{2}{n^{3/8}} n(n-1) < \frac{2}{n} \).
\[ P \left[ \frac{A'B'+B'+C+D \leq 0}{E_2} \right] + P \left[ \frac{A'C+D \geq 0}{E_2} \right] < 1. \]

\[
P(E_1) + P(E_2) = \frac{1}{2} P(E_1 | A = -1) + \frac{1}{2} P(E_1 | A = +1) \\
+ \frac{1}{2} P(E_2 | A = -1) + \frac{1}{2} P(E_2 | A = +1)
\]

\[
= \frac{1}{2} \left( P(E_1 | A = -1) + P(E_2 | A = -1) \right) + \left[ P(E_1 | A = +1) + P(E_2 | A = +1) \right]
\]

Since \( \frac{S_1 + S_2}{2} < 1 \), we need \( S_1 < 1 \) or \( S_2 < 1 \).

Now compute \( S_1 \) and \( S_2 \); then choose the value of \( A \) that minimizes the sum.

\[
P(E_1 | A = -1) = P(C+D \leq 0) = \frac{3}{4}
\]

\[
P(E_2 | A = -1) = P(-C+D > 0) = P(C+D > 0) = \frac{1}{4}
\]

Hence \( S_1 = \frac{3}{4} + \frac{1}{4} = 1 \).

\[
P(E_1 | A = +1) = P(2B+C+D \leq 0) = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}
\]

\[
P(E_2 | A = +1) = P(C+D > 0) = \frac{1}{8}
\]

Hence \( S_2 = \frac{5}{8} + \frac{1}{8} = \frac{7}{8} \).

Since \( S_2 < S_1 \), choose \( A = +1 \).

Fix \( i, j \) \& \( i \neq j \) let \( k \) be the third index, i.e.

\[
\{ i, j, k \}. \quad \text{Fix } S \subseteq A_i \cup A_j \cup A_k.
\]

\[
S \subseteq A_i \cup A_j \cup A_k.
\]

\[
P \left( \text{all edges from } S \text{ fall in } S' \cup A_i \cup A_k \right)
\]

\[
< \left[ \frac{3^k + 2n}{3n} \frac{3^k + 2n - 1}{3n - 1} \ldots \frac{3^k + 2n - 5 + 1}{3n - 5 + 1} \right]
\]

\[
< \left( \frac{S^k + 2n}{3n} \right)^3 d
\]
Applying Erdős's inequality for all choices of $i, j, s, s'$

$$P\text{ (failure) } < 6 \left(\frac{m}{n}\right) \left(\frac{m'}{n'}\right) \left(\frac{k^2 + 2n}{3n}\right).$$

If we make $s' = \frac{n}{2}$ and large, $P\text{ (failure) }$ becomes less than 1.

Hence $P\text{ (the given properties are satisfied) } > 0$.

Hence a graph with the given properties exists.