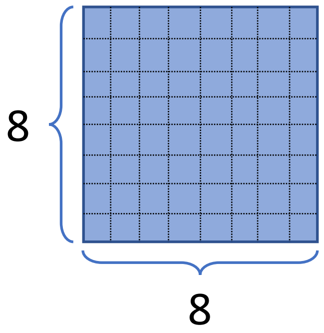
Basis Functions and Sparsity: What can happen in an 8x8 image window?



Theoretically, 256^{64} possible images

But, which ones happen?

How to represent images?

- Basis Functions / Fourier Series
- Overcomplete bases, sparse coding
- Learning bases: (i) PCA, (ii) Sparsity, (iii) Matched Filters

Representing images in terms of basis function

Classic: Orthogonal set of basis functions

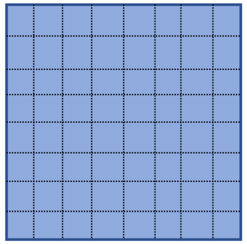
$$\{b_i(x) : i = 1, \ldots, N\}$$

where $\sum_x \{b_i(x)\}^2 = 1$

$$\sum_x b_i(x) b_j(x) = 0, \text{ if } i \neq j$$

or $\int dx \{b_i(x)\}^2 = 1$

$$\int dx\, b_i(x) b_j(x) = 0, \text{ if } i \neq j$$



$\mathcal{D}$

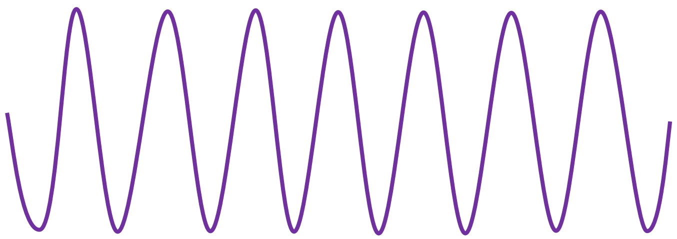
8x8 patch

Examples

- Sinusoids / Fourier Analysis
- Haar Bases
- Impulse Function

JPEG Coding

Choose basis function to be sinusoids



Represent image by $I(x) = \sum_i \alpha_i b_i(x)$

because the bases are orthonormal, we can solve to get

$$\alpha_i = \sum_x I(x) b_i(x) \quad (\text{or} \int dx \cdots)$$

Image represented by the coefficients $\{\alpha_i\}$

Also we could minimize an error $\sum_x \left| I(x) - \sum_i \alpha_i b_i(x) \right|^2$

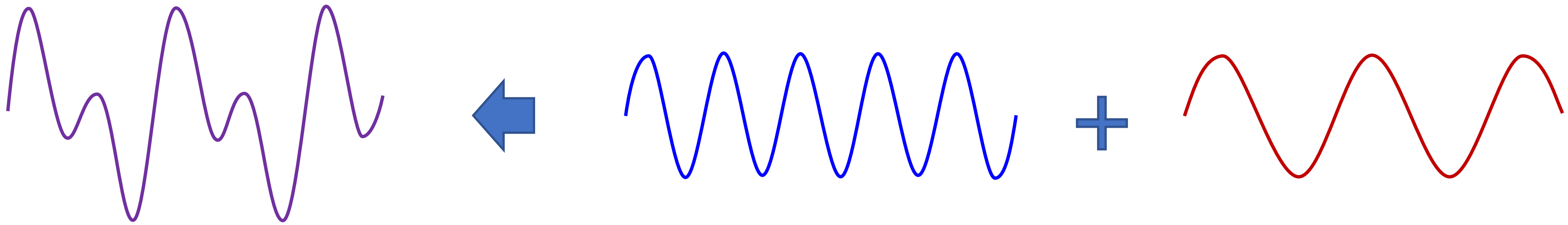
And try to restrict the number of non-zero α's

This gives standard image format of JPEG if we use sinusoids

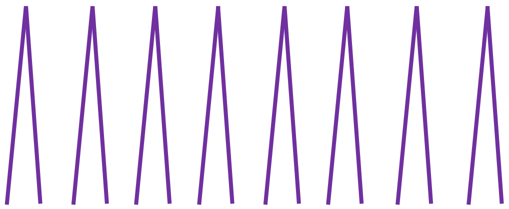
Sinusoids / Fourier Theory work well

if the image can be approximated well by a set of sinusoids

E.G.

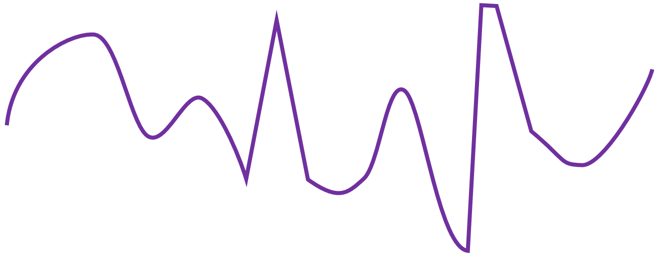


But an image like this:



is better approximated by
a set of impulse functions

And an image like this:



Is badly modeled by either

Over-complete Bases

Represent the image by an over-complete set

E.G. all the sinusoids and all the impulse functions. Represent the image by a combination of sinusoids and impulses.

But now we have a problem

There will be many ways to represent the image in form

$$I(x) = \sum_i \alpha_i b_i(x)$$

because we could represent it by sinusoids only, or by impulse function only, or by combinations

Sparsity L1-Sparsity

Determine the α's by minimizing

L1-norm

$$E[\alpha] = \sum_x \left\{ I(x) - \sum_i \alpha_i b_i(x) \right\}^2 + \underbrace{\lambda \sum_i \boxed{|\alpha_i|}}_{\text{regularization}}$$

Note: $E[\alpha]$ is a convex function (L1-norm is convex

- There are efficient algorithms to estimate $\hat{\alpha} = \arg\min E[\alpha]$
- Solution: $I(x) = \sum_i \hat{\alpha}_i b_i(x)$

By a "miracle" (later in lecture), many of the α's will be zero

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Extreme Sparsity: Matched Filters

Set of basis function: $\{b_i(x)\}$

Represent each image by one basis function only

$$E[\alpha] = \sum_x \left| I(x) - \sum_i \alpha_i b_i(x) \right|^2 \quad \text{with constant only one} \quad \alpha_i \neq 0$$

Algorithm estimate $\hat{\alpha} = \arg\min E[\alpha]$

Set $\hat{\alpha}_i = \arg\min \sum_x \left| I(x) - \alpha_i b_i(x) \right|^2 = \arg\min \sum_x I(x) b_i(x) \Leftarrow \sum \{b_i(x)\}^2 = 1$

Choose $\hat{i} = \min_i \sum_x \left| I(x) - \hat{\alpha}_i b_i(x) \right|^2 \Rightarrow$ Set $\alpha_{\hat{i}} = \hat{\alpha}_i$

$\alpha_j = 0$ otherwise

Comments

We described three ways to represent images using basis functions

- Classical: e.g. Fourier Theory / Harr Basis
- L1-Sparsity
- Matched Filters

Both, overcomplete

But what bases to use?

- We can use the bases, like sinusoids (20th century math)
- Or we can learn them from data (21th century math)

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Learning the bases

Let's start with the classical approach

Bases are orthogonal → $\sum_x b_i(x) b_j(x) = S_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ (Kronecker Delta)

Dataset of images: $\{ I^\mu(x) : \mu \in \Lambda \}$

Energy Function $E[b,\alpha] = \frac{1}{|\Lambda|} \sum_{\mu \in \Lambda} \sum_x \left\{ I^\mu(x) - \sum_i \alpha_i^\mu b_i(x) \right\}^2$

Note: basis functions are the same for all images

the coefficients α_i^μ vary between images

Minimize

$$E[b,\alpha] = \frac{1}{|\Lambda|}\sum_{\mu\in\Lambda}\sum_{x}\left\{I^{\mu}(x) - \sum_{i}\alpha_i^{\mu} b_i(x)\right\}^2$$

w.r.t. (b,α)

This is simply Principal Component Analysis (PCA)

Provided we extract the means from the images

$$I^{\mu}(x) \to I^{\mu}(x) - \frac{1}{|\Lambda|}\sum_{\mu\in\Lambda} I^{\mu}(x)$$

so that $\sum_{\mu} I^{\mu}(x) = 0$

(after subtraction)

Solution: Singular Value Decomposition (SVD) implies that

The basis function $b_i(x)$ are the eigenvectors of the correlation matrix

$$K(x,y)=\frac{1}{|\Lambda|}\sum_{\mu\in\Lambda} I^{\mu}(x)I^{\mu}(y)$$

The coefficients $\alpha_i^{\mu}=\sum_x b_i(x)I^{\mu}(x)$ (as before)

We can restrict the number of basis function by only use those eigenvectors whose eigenvalues are above a threshold T

$$\Rightarrow \sum_y K(x,y)b_i(y)=\lambda_i b_i(x), \quad \text{keep } b_i(x) \text{ if } \lambda_i > T$$

What are the eigenvectors of image patches?

Claim If the image patches are randomly drawn from real images, then the eigenvectors are sinusoids?

Why? Because images are shift-invariant

$$K(x, y) = F(x - y)$$

The correlation function depends only on the different (x-y)

Eigenvectors: $\sum_y F(x-y)e(y) = \lambda e(x)$

Sinusoids ➔ proof: apply the convolution theorem

So PCA doesn't help much

You know you will get sinusoids before you look at the images

It is different if we align the images

For example, if we have images of faces and center them in the image patch

The alignment means that we remove shift-invariance

But it is not possible to align general images

Now try sparsity – Olshausen & Field, 1996

$$E[b,\alpha] = \frac{1}{|\Lambda|}\sum_{\mu\in\Lambda}\sum_{x}\left\{I^{\mu}(x) - \sum_{i}\alpha_i^{\mu} b_i(x)\right\}^2 + \lambda\sum_{\mu\in\Lambda}\sum_{i}\left|\alpha_i^{\mu}\right|$$

constraint: $\sum\{b_i(x)\}^2 = 1$

Minimize E w.r.t. (b,α)

Note: $E[b,\alpha]$ is convex in α if b is fixed (sparsity)
$E[b,\alpha]$ is convex in b if α is fixed

Alternative Algorithm

- Initialize b's
- Minimize w.r.t a and b alternatively
- Guaranteed to converge to local minima

code available online

Olshausen & Field, 1996

Applied these to natural images (See examples)

This gives more interesting bases than PCA

Note: Deep Neural Networks obtain similar bases

Final Alternative Matched Filters $\sum \{b_i(x)\}^2 = 1$

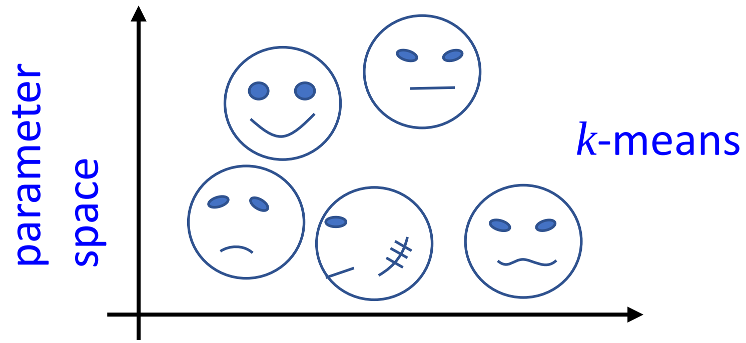
Minimize $E[b,\alpha] = \dfrac{1}{|\Lambda|} \sum_{\mu \in \Lambda} \sum_x \left\{ I^\mu(x) - \sum_i \alpha_i^\mu b_i(x) \right\}^2$

with constraint that only one α_i^μ
is non-zero for each μ

How to minimize?

➡ Convert this to k-means clustering

Requires normalizing each image $I^\mu(x) \to \dfrac{I^\mu(x)}{\sqrt{\sum_x \{I^\mu(x)\}^2}}$ so that $\sum_x \{I^\mu(x)\}^2 = 1$



➡ Implies that the best $\alpha_i^\mu = 1$

The Miracle of Sparsity

Sparsity represents an input y by

$$\hat{\alpha} = \arg\min \left\{ \left| y - \sum_i \alpha_i b_i \right|^2 + \lambda \sum_i |\alpha_i| \right\}$$

The miracle: many $\hat{\alpha}_i$ will be zero ➡ Why?

This won't happen if we replaced $\sum_i |\alpha_i|$ (L^1-loss) by $\sum_i \alpha_i^2$ (L^2-loss)

(Easy to see, with L2-loss you can compute $\hat{\alpha}$ analytically)

Why the miracle? 1D case

Let $f(a;x) = (x-a)^2 + \lambda |a|$

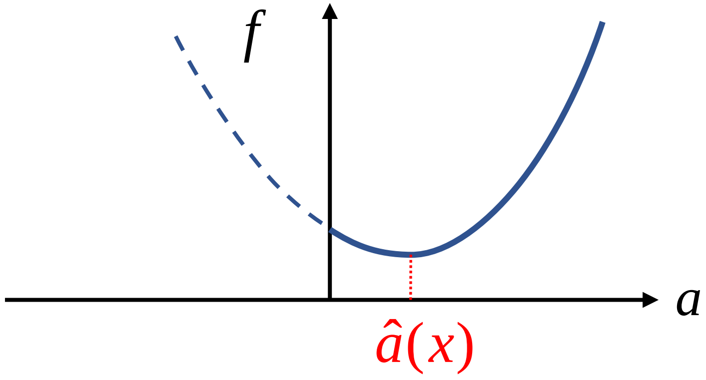
Claim $\hat{a}(x) = x - \lambda/2, \text{ if } x \geq \lambda/2$

$\hat{a}(x) = x + \lambda/2, \text{ if } x \leq -\lambda/2$

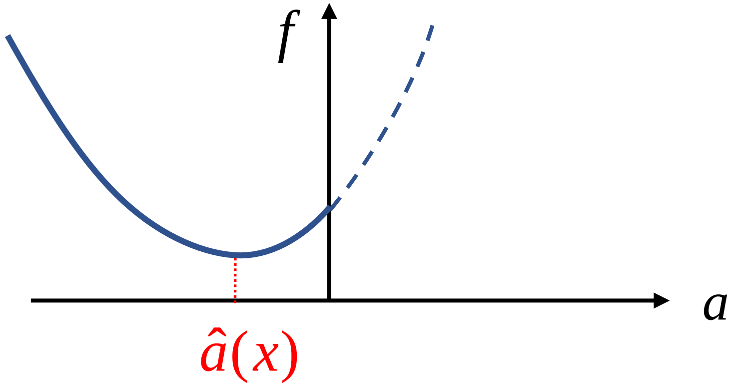
$\hat{a}(x) = 0, \quad , \text{ if } |x| \leq \lambda/2$

here $\hat{a}(x) = \arg\min f(a;x)$

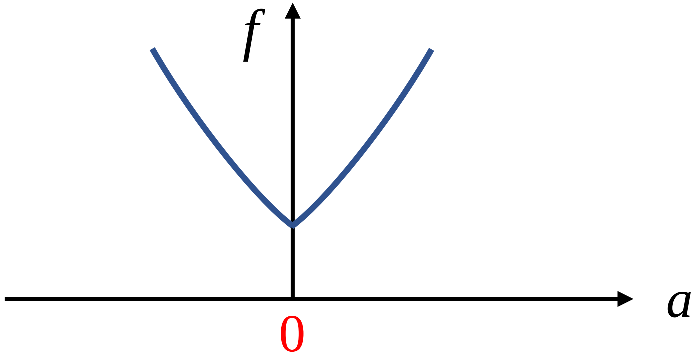
if $x \geq \lambda/2$



if $x \leq -\lambda/2$



if $|x| \leq \lambda/2$



Can check analytically

If $a \geq 0$ $\quad f_+(a;x) = (x-a)^2 + \lambda a$

$$\frac{df_+}{da} = -2(x-a) + \lambda$$

minima at $\hat{a} = x - \lambda / 2$

but $\hat{a} \geq 0 \Rightarrow x \geq \lambda / 2$

Similarly, If $a \leq 0$ $\quad f_-(a;x) = (x-a)^2 - \lambda a$

$$\frac{df_-}{da} = -2(x-a) - \lambda$$

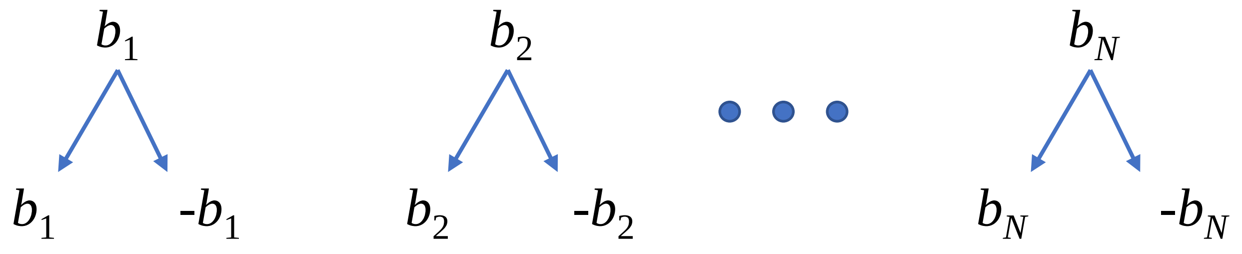
minima at $\hat{a} = x + \lambda / 2$

but $\hat{a} \leq 0 \Rightarrow x \geq -\lambda / 2$

In higher dimensions

Reformulate the problem in terms of convex hulls

First, duplicate each basis function



new $2N$ bases: $\overline{b}_i$

Then we can express $\sum_{i=1}^{N} \alpha_i b_i = \sum_{i=1}^{2N} \overline{\alpha}_i \overline{b}_i$ with $\overline{\alpha}_i \geq 0$

Trick
$$\begin{aligned} \alpha_i b_i &= \alpha_i b_i, & \text{if } \alpha_i \geq 0 \\ &= (-\alpha_i)(-b_i), & \text{if } \alpha_i < 0 \end{aligned}$$

In higher dimensions

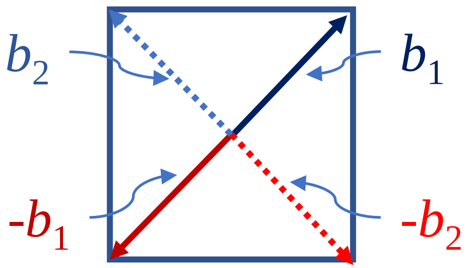
Now consider encoding an input y

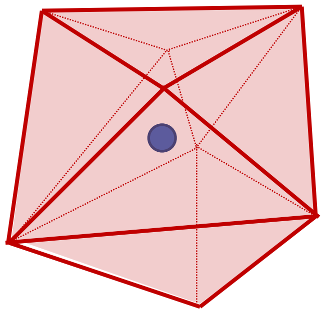
$$\hat{\bar{\alpha}} = \arg\min\left\{\left|y-\sum_i \bar{\alpha}_i b_i\right|^2 + \lambda\sum_i \bar{\alpha}_i\right\}, \quad \text{s.t. } \bar{\alpha}_i \geq 0$$

Let $\quad \sum_{i=1}^{2N} \bar{\alpha}_i = \alpha$

Then $\quad \left\{ y : \left\| y - \sum_i \bar{\alpha}_i \bar{b}_i \right\| \quad s.t. \sum_i \bar{\alpha}_i = \alpha \right\}$ specifies the convex hull of the $\{\bar{b}_i\}$ with radius α

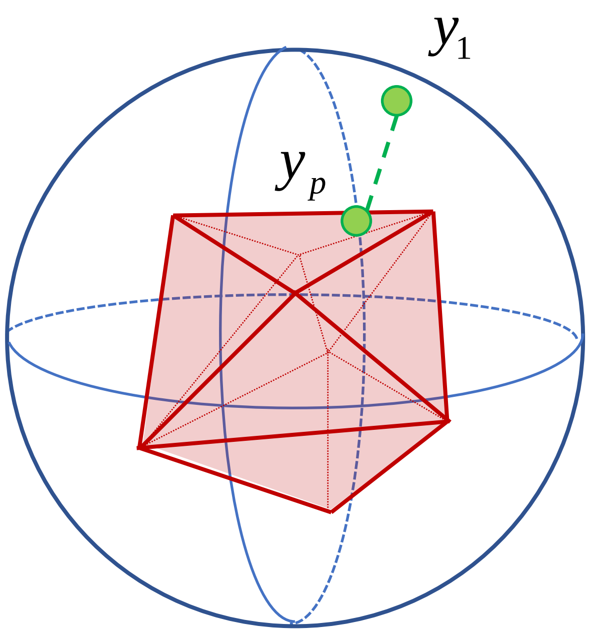
E.G.





In higher dimensions

Consider an input data y, w.l.o.g. $|y| = 1$ ⬅ Lies on a sphere



Hence, solving for $\bar{\alpha}_i$ corresponds to finding the closest point y_p on the convex hull

Sparsity ➔find closest point on convex hull while penalizing the radius α of the convex hull

Hence, y is projected to a point y_p on the boundary of the convex hull

In higher dimensions

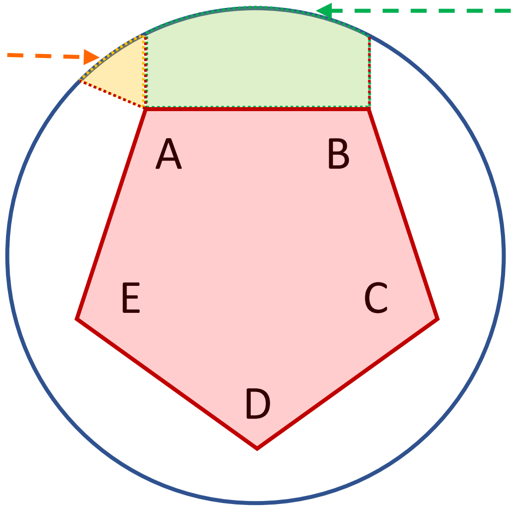
Increasing the size of λ

Corresponds w increasing the penalty for the radius of the convex hull

Hence causing the radius to get smaller

Where do point project?

Projected to basis A

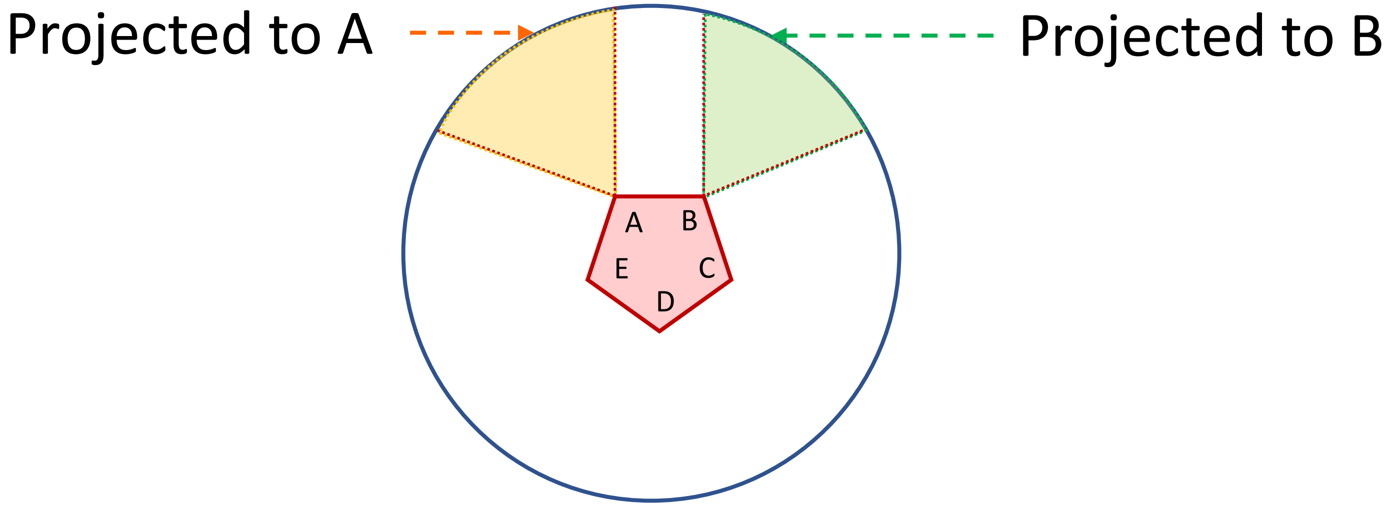


Projected to bases A&B
(zero coefficients for C, D, and E)

This shows that many bases will have zero coefficients

In higher dimensions, Increasing the size of λ

As λ gets bigger, the convex hull gets smaller and increasingly bases have non-zero coefficients



This gives geometric intuition into the miracle of sparsity