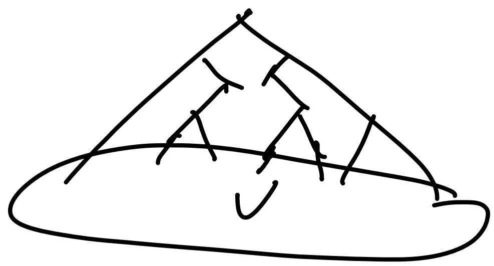
FRT Tree Embedding

Def: A tree metric (V', T) for a set of nodes V is a tree T with vertices V' s.t. $V \subseteq V'$ are the leaves of T, and a nonnegative length for each edge in T



So tree metric for V is tree with V as leaves, gives distances between leaves

Thm [Fakcharoenphol, Rao, Talwar '03]:

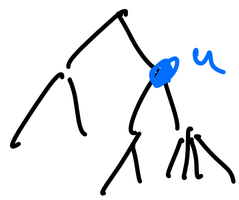
Let (V, d) be a metric space. Then there is a randomized, polytime algorithm that produces a tree metric (V', T) for V s.t.

1) $d(u,v) \leq d_T(u,v) \quad \forall u,v \in V$
2) $E[d_T(u,v)] \leq O(\log n) \cdot d(u,v) \quad \forall u,v \in V$

Embedding into a distribution of trees (dominating trees)

Setup:

- $B(u,r) = \{v \in V : d(u,v) \leq r\}$
- WLOG, $\min_{u,v \in V : u \neq v} d(u,v) = 1$ (rescaling)
- For $S \subseteq V$, $\text{diam}(S) = \max_{u,v \in S} d(u,v)$
- $\Delta = 2^{\lceil \log \text{diam}(V) \rceil}$ (smallest power of 2 s.t. $\Delta \geq \text{diam}(V)$)



Hierarchical Cut Decomposition:

Tree metric (V', T) on V s.t.

1) $\forall u \in V'$, $S_u = \{v \in V : v \text{ descendent of } u \text{ in } T\}$

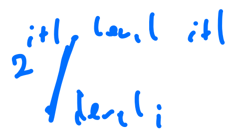
   $\Rightarrow x \in V$, $S_x = \{x\}$

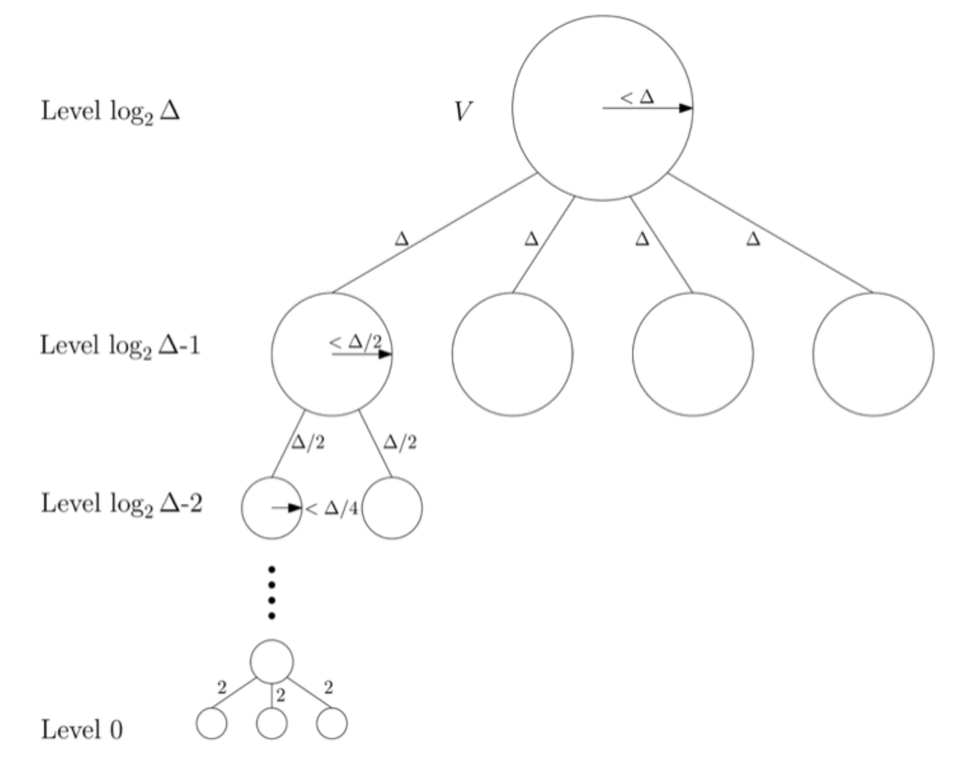
   $S_r = V$ (r root of T)

2) If u at level i (height i) in T,

   $\text{diam}(S_u) < 2^i$

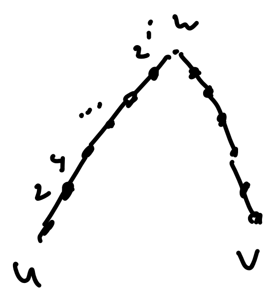
3) Length on an edge between level i and level $i+1$ is 2^{i+1}





Lemma: If the LCA of $u,v \in V$ at level i, then $d_T(u,v) \le 2^{i+2}$. Also, $d_T(u,v) \ge d(u,v)$ $\forall u,v \in V$

Pf: Let $u,v \in V$, $w = LCA(u,v)$ at level i



$$d_T(u,w) = \sum_{j=1}^{i} 2^j = 2^{i+1} - 2$$

$$d_T(v,w) = \sum_{j=1}^{i} 2^j = 2^{i+1} - 2$$

$$\Rightarrow 2^i \le d_T(u,v) \le 2^{i+2}$$

$$\Rightarrow d_T(u,v) \le 2^{i+2} \checkmark$$

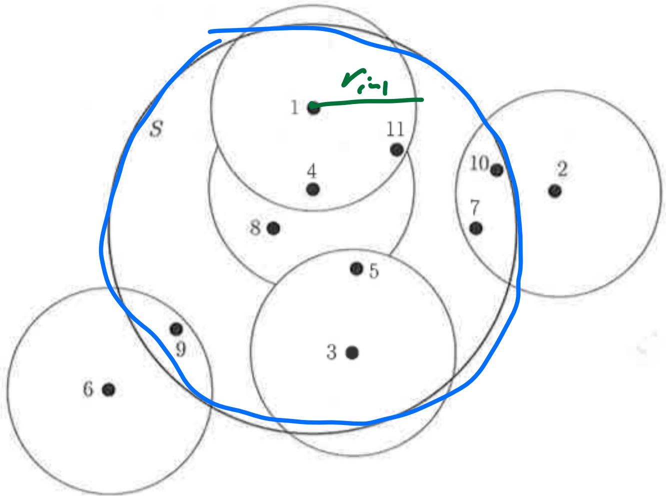
$$d(u,v) \le diam(S_w) \le 2^i \le d_T(u,v) \checkmark$$

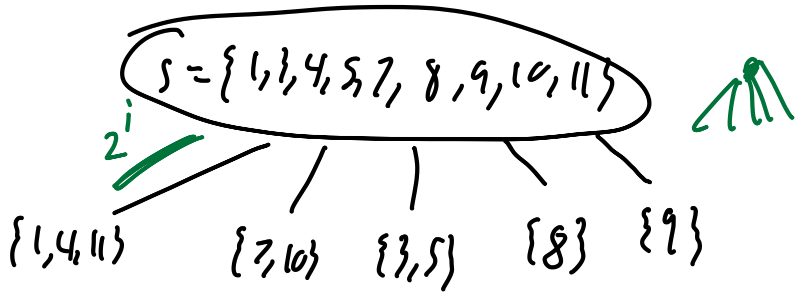
Note: Hierarchical cut decompositions very special tree metrics:

Hierarchically Well-Separated Tree (HST)

FRT Algorithm:

- Sample r_0 uniformly at random from $[\frac{1}{2}, 1)$
- Let $r_i = 2^i r_0$
- Sample a permutation π of V uniformly at random
- Initialize T to be only a root node r at level $\log \Delta$, $S_r = V$
- for $i = \log \Delta - 1$ down to 0:
    - Let $\mathcal{C}$ be nodes at level $i+1$
    - for $C \in \mathcal{C}$:
        - Let $S = C$
        - for $j = 1$ to n:
            - $P = B(\pi(j), r_{i-1}) \cap S$
            - if $P \neq \emptyset$:
                - Add P as child of C at level i
                - $S = S \setminus P$





Claim: Hierarchical Cut Decomposition

Pf: Let u at level i. Need to show $\text{diam}(S_u) \le 2^i$

$$S_u = B(\pi(j), r_{i-1}) \cap S \quad \text{for some } S, j$$

$$\Rightarrow \text{diam}(S_u) \le \text{diam}(B(\pi(j), r_{i-1}))$$

$$\le 2 \cdot r_{i-1} = 2 \cdot r_0 2^{i-1} < 2 \cdot 2^{i-1} = 2^i \checkmark$$

$\Rightarrow$ Lemma implies $d(u,v) \le d_T(u,v)$

Thm: $E[d_T(u,v)] \le O(\log n)\, d(u,v) \quad \forall u,v \in V$

Rest of class proving this.

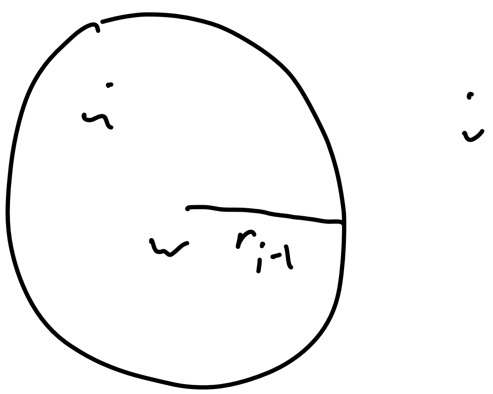
Fix $u,v \in V$

Def: - w settles u,v at level i if w is the first vertex in π s.t. $B(w, r_{i-1}) \cap \{u,v\} \neq \emptyset$

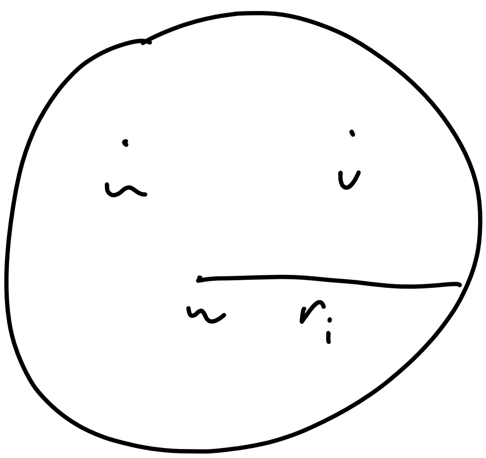
- w cuts u,v at level i if $|B(w, r_{i-1}) \cap \{u,v\}| = 1$

Observation: $LCA(u,v)$ is at level $i+1$ if i is the largest value s.t. the vertex w which settles u,v at level i also cuts u,v at level i

Pf sketch:



u,v not in same level i cluster



u,v in same level $i+1$ and above clusters

Useful Random Variables:

$$S_{iw} = \begin{cases} 1 & \text{if } w \text{ settles } u,v \text{ at level } i \\ 0 & \text{otherwise} \end{cases}$$

$$X_{iw} = \begin{cases} 1 & \text{if } w \text{ cuts } u,v \text{ at level } i \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ there is a node which both settles and cuts u,v at level i iff $\sum_{w \in V} S_{iw} X_{iw} = 1$

Let i^* be level of $LCA(u,v)$

$\Rightarrow$ 1) $d_T(u,v) \leq 2^{i^*+2}$, and

2) i^*-1 largest i s.t. $\sum_{w \in V} S_{iw} X_{iw} = 1$

$$\Rightarrow d_T(u,v) \leq 2^{i^*+2} = \max_{i : \sum_{w \in V} S_{iw} X_{iw} = 1} 2^{i+3}$$

$$\leq \sum_{i=0}^{\log \Delta} 2^{i+3} \sum_{w \in V} S_{iw} X_{iw}$$

$$= \sum_{i=0}^{\log \Delta} \sum_{w \in V} 2^{i+3} S_{iw} X_{iw}$$

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Take expectations:

$$E[d_T(u,v)] \leq E\left[\sum_{w \in V} \sum_{i=0}^{\log \Delta} 2^{i+3} S_{iw} X_{iw}\right]$$

$$= \sum_{w \in V} \sum_{i=0}^{\log \Delta} 2^{i+3} E[S_{iw} X_{iw}]$$

$$= \sum_{w \in V} \sum_{i=0}^{\log \Delta} 2^{i+3} Pr[S_{iw} = 1 \wedge X_{iw} = 1]$$

$$= \sum_{w \in V} \sum_{i=0}^{\log \Delta} 2^{i+3} Pr[S_{iw} = 1 \mid X_{iw} = 1] \cdot Pr[X_{iw} = 1]$$

Intuition: X_{iw} depends only on r_0, not π.

Hopefully easier to analyze

Two lemmas, show how they imply main theorem

$E[d_T(u,v)] \leq O(\log n)\, d(u,v)$, then prove lemmas

<u>Lemma 1</u>: For every $w \in V$, there is some $b_w \in \mathbb{R}_{\geq 0}$ s.t.

1) $Pr[S_{iw} = 1 \mid X_{iw} = 1] \leq b_w \;\; \forall i$, and

2) $\sum_{w \in V} b_w \leq O(\log n)$

Lemma 2: $\sum_{i=0}^{\log\Delta} 2^{i+3} \Pr[X_{iw}=1] \le 32\, d(u,v) \quad \forall w \in V$

Combine previous inequalities:

$$E[d_T(u,v)] \le \sum_{w\in V} \sum_{i=0}^{\log\Delta} 2^{i+3} \Pr[S_{iw}=1 \mid X_{iw}=1] \cdot \Pr[X_{iw}=1]$$

$$\le \sum_{w\in V} b_w \sum_{i=0}^{\log\Delta} 2^{i+3} \Pr[X_{iw}=1]$$

$$\le \sum_{w\in V} b_w \, 32\, d(u,v)$$

$$= 32\, d(u,v) \sum_{w\in V} b_w$$

$$\le O(\log n)\cdot d(u,v)$$

Pf of Lemma 1:

want to analyze $\Pr[S_{iw}=1 \mid X_{iw}=1]$.

Order V by distance to $\{u,v\}$:

$$d(w_j, \{u,v\}) \le d(w_{j+1}, \{u,v\}) \quad \forall j$$

Sps $X_{iw_j}=1 \Rightarrow |B(w_j, r_{i-1}) \cap \{u,v\}| = 1$

$\Rightarrow |B(w_k, r_{i-1}) \cap \{u,v\}| \geq 1 \quad \forall\, k<j$

$\Rightarrow$ if w_k earlier in π than w_j, $S_{iw_j} = 0$

$\Pr[w_j \text{ earlier than } w_k \text{ in } \pi \ \forall\, k<j] =$

$= \frac{1}{j}$ (random permutation)

Set $b_{w_j} = \frac{1}{j}$

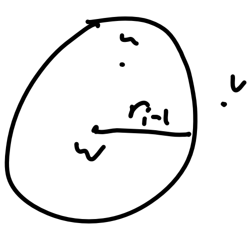
$\Rightarrow \Pr[S_{iw_j}=1 \mid X_{iw_j}=1] \leq \frac{1}{j} = b_{w_j}$

$$\sum_{w \in V} b_w = \sum_{j=1}^{n} b_{w_j} = \sum_{j=1}^{n} \frac{1}{j} = H_n = O(\log n)$$

Pf of Lemma 2:

WTS: $\sum_{i=0}^{\log \Delta} 2^{i+3} \Pr[X_{iw}=1] \leq 32\, d(u,v)$

WLOG, $d(w,u) \leq d(w,v)$

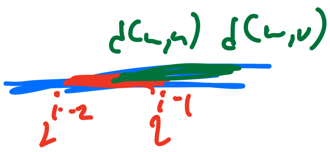
$X_{iw}=1 \Leftrightarrow$ 

$d(w,u) \leq r_{i-1} \leq d(w,v)$

r_{i-1} distributed <u>uniformly</u> in $[2^{i-2}, 2^{i-1})$

$$\Rightarrow \Pr[X_{iw} = 1] = \Pr[r_{i-1} \in [d(w,u), d(w,v))]$$

$$= \frac{|[2^{i-2}, 2^{i-1}) \cap [d(w,u), d(w,v))|}{|[2^{i-2}, 2^{i-1})|}$$



$$= \frac{|[2^{i-2}, 2^{i-1}) \cap [d(w,u), d(w,v))|}{2^{i-2}}$$

$$\Rightarrow 2^{i+3} \Pr[X_{iw} = 1] = \frac{2^{i+3}}{2^{i-2}} |[2^{i-2}, 2^{i-1}) \cap [d(w,u), d(w,v))|$$

$$= 32\, |[2^{i-2}, 2^{i-1}) \cap [d(w,u), d(w,v))|$$

$$\Rightarrow \sum_{i=0}^{\log \Delta} 2^{i+3} \Pr[X_{iw} = 1] \leq$$

$$\leq \sum_{i=0}^{\log \Delta} 32\, |[2^{i-2}, 2^{i-1}) \cap [d(w,u), d(w,v))|$$

$$= 32\, |[d(w,u), d(w,v))|$$

$$= 32\,(d(w,v) - d(w,u))$$

$$\leq 32\, d(u,v) \qquad (d(w,v) \leq d(w,u) + d(u,v))$$