Multiway Cut:

Input: - $G = (V, E)$ (undirected)

- costs $c: E \to \mathbb{R}^+$

- $T = \{s_1, s_2, \ldots, s_k\} \subseteq V$

Feasible: $A \subseteq E$ s.t. $G \setminus A$ has no $s_i - s_j$ path $\forall i \neq j \in [k]$

Objective: $\min \sum_{e \in E} c(e)$

Last class: 2-approximation, $2(1-\frac{1}{k})$ integrality gap.

Trying to do better via new LP

Think of opt cut $F \subseteq E$.

Let $C_i = \{w \in V : w$ reachable from s_i in $G \setminus F\}$

$\Rightarrow C_i \cap C_j = \emptyset \quad \forall i \neq j \in [k]$

<u>Claim</u>: C_i's form partition in optimal solution

New LP:

Variables: $x_u^i = \begin{cases} 1 & \text{if } u \in C_i \\ 0 & \text{otherwise} \end{cases}$

$z_e^i = \begin{cases} 1 & \text{if } e \in \delta(C_i) = E(C_i, \bar{C_i}) \\ 0 & \text{otherwise} \end{cases}$

$$\min \frac{1}{2} \sum_{e \in E} \sum_{i=1}^{k} c(e) z_e^i \qquad \sum_{e \in E} c(e) \left(\frac{1}{2} \sum_{i=1}^{k} z_e^i \right)$$

s.t.

$$\sum_{i=1}^{k} x_u^i = 1 \qquad \forall u \in V$$

$$x_{s_i}^i = 1 \qquad \forall i \in [k]$$

$$z_e^i \geq x_u^i - x_v^i \qquad \forall e = \{u,v\} \in E,\ \forall i \in [k]$$

$$z_e^i \geq x_v^i - x_u^i \qquad \forall e = \{u,v\} \in E,\ \forall i \in [k]$$

$$0 \leq x_u^i \leq 1 \qquad \forall u \in V,\ \forall i \in [k]$$

$$0 \leq z_e^i \leq 1 \qquad \forall e \in E,\ \forall i \in [k]$$

Thm: ILP version is exact formulation

Pf: Exercise

Rewrite in more compact/intuitive way

$\|x-y\|_2 = \sqrt{\sum_{i=1}^{k}(x_i-y_i)^2}$

Def: For $x, y \in \mathbb{R}^k$, $\|x-y\|_1 = \sum_{i=1}^{k} |x^i - y^i|$,

where x^i is i'th coordinate of x

y^i is i'th coordinate of y

Def: k-simplex $\Delta_k = \{x \in \mathbb{R}^k : \sum_{i=1}^{k} x^i = 1 \text{ and } x^i \geq 0 \ \forall i\}$

Def: e_i is vector with 1 in coordinate i,
0 in all other coordinates

Given LP solution (x, z), $\forall u$ let $x_u = (x_u^1, x_u^2, \dots, x_u^k)$

In any optimal LP solution, $z_e^i = |x_u^i - x_v^i|$

$e = \{u, v\}$

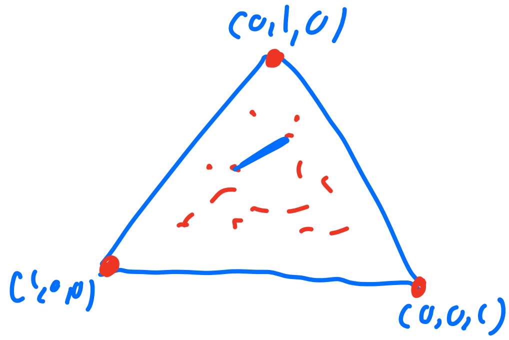
$\Rightarrow \|x_u - x_v\|_1 = \sum_{i=1}^{k} z_e^i$

Rewritten LP:

$$\min \frac{1}{2} \sum_{\substack{e \in E \\ e=\{u,v\}}} c(e) \|x_u - x_v\|_1 = \frac{1}{2} \sum_{e \in E} c(e) \sum_{i=1}^{k} z_e^i$$

$$\text{s.t.} \quad x_{s_i} = e_i \qquad \forall i \in [k]$$

$$x_u \in \Delta_k$$



Not technically LP, but equivalent to longer LP

In other words: embed vertices into k-simplex, terminals at corners, min weighted ℓ_1-distances!

$\Rightarrow$ integral solution: all nodes at corners

Note: $\|x_{s_i} - x_{s_j}\|_1 = 2$

<u>Def</u>: $B(s_i, r) = \{v \in V : \|e_i - x_v\|_1 \leq r\}$

<u>Rounding Algorithm</u> [Calinescu, Karloff, Rabani]:

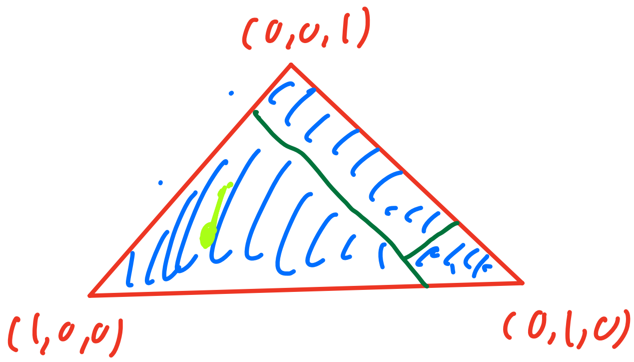
- Initially $C_i = \emptyset \quad \forall i \in [k]$

  $X = \emptyset$
- Pick $r \in [0, 2)$ uniformly at random
- Pick π a permutation of $[k]$ uniformly at random
- for $i = 1$ to $k-1$
  - $C_{\pi(i)} = B(s_{\pi(i)}, r) \setminus X$
  - $X = X \cup C_{\pi(i)}$

- Set $C_{\pi(k)} = V \setminus X$

- Return $A = \bigcup_{i=1}^{k} \delta(C_i)$

Basically "one level" of FRT!



$\pi = (1,3,2)$

<u>Thm</u>: CKR is a $\frac{3}{2}$-approximation

Random variable: $Z_{uv} = \begin{cases} 1 & \text{if } u,v \text{ separated by cut} \\ 0 & \text{otherwise} \end{cases}$

<u>Claim</u>: $\Pr[Z_{uv} = 1] \leq \frac{3}{4} \|x_u - x_v\|_1 \quad \forall \{u,v\} \in E$

Assuming claim:

$$E[ALG] = E\left[\sum_{e=\{u,v\} \in E} c(e) Z_{u,v}\right]$$

$$= \sum_{e=\{u,v\} \in E} c(e)\, E[Z_{uv}]$$

$$\leq \sum_{e=\{u,v\}\in E} c(e)\,\frac{3}{4}\,\|x_u - x_v\|_1$$

$$= \frac{3}{2}\cdot\frac{1}{2}\cdot\sum_{e=\{u,v\}\in E} c(e)\,\|x_u - x_v\|_1$$

$$= \frac{3}{2}\cdot LP \leq \frac{3}{2}\cdot OPT$$

<u>Pf of claim</u>:

$$\Pr[Z_{uv} = 1] \leq \frac{3}{4}\,\|x_u - x_v\|_1 \quad \forall \{u,v\}\in E$$

Start like FRT. Fix $e=\{u,v\}\in E$.

$$S_i = \begin{cases} 1 & \text{if } i \text{ first index in } \pi \text{ s.t. } B(s_i, r)\cap\{u,v\}\neq\emptyset \\ 0 & \text{otherwise}\end{cases}$$

s_i settles u,v

$$X_i = \begin{cases} 1 & \text{if } |B(s_i, r)\cap\{u,v\}| = 1 \\ 0 & \text{otherwise}\end{cases}$$

s_i cuts u,v

<u>Observation</u>: $Z_{uv} = \sum_{i=1}^{k} S_i X_i$

(one i settles, separated if it also cuts)

$$\Rightarrow \Pr[Z_{uv}=1] = E[Z_{uv}] = E\left[\sum_{i=1}^{k} S_i X_i\right] = \sum_{i=1}^{k} E[S_i X_i]$$

$$= \sum_{i=1}^{k} \Pr[S_i = 1 \wedge X_i = 1]$$

$$= \sum_{i=1}^{k} \Pr[S_i = 1 \mid X_i = 1] \cdot \Pr[X_i = 1]$$

useful Lemma: relate distances to single coordinate

Lemma: $\|e_i - x_u\|_1 = 2(1 - x_u^i) \quad \forall u \in V,\ i \in [k]$

Pf: $\|e_i - x_u\|_1 = \sum_{j=1}^{k} |e_i^j - x_u^j|$

$$= 1 - x_u^i + \sum_{j \neq i} x_u^j$$

$$= 1 - x_u^i + (1 - x_u^i) \qquad \left(\sum_{j=1}^{k} x_u^j = 1\right)$$

$$= 2(1 - x_u^i)$$

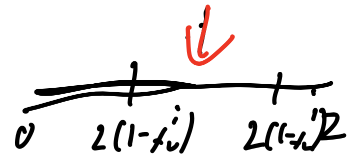
WLOG, $x_u^i \geq x_v^i$

So $\Pr[X_i = 1] = \Pr[\,|B(s_i, r) \cap \{u, v\}| = 1\,]$

$$= \Pr[\min(\|e_i - x_u\|_1, \|e_i - x_v\|_1) \leq r < \max(\|e_i - x_u\|_1, \|e_i - x_v\|_1)]$$

$$= \Pr[\min(2(1-x_u^i), 2(1-x_v^i)) \leq r < \max(2(1-x_u^i), 2(1-x_v^i))]$$

$$= Pr[2(1-x_u^i) \le r < 2(1-x_v^i)]$$



$$= \frac{2(1-x_v^i) - 2(1-x_u^i)}{2}$$

r uniformly distributed in (0, 2)

$$= x_u^i - x_v^i$$

$$= |x_u^i - x_v^i|$$

Let $\ell = \underset{i \in [k]}{\arg\min} \min(1 - x_u^i, 1 - x_v^i)$

S_ℓ terminal closest to $\{u, v\}$ in ℓ_1-metric

$$\Rightarrow Pr[S_\ell = 1 \mid X_\ell = 1] \cdot Pr[X_\ell = 1] \le Pr[X_\ell = 1]$$

$$= |x_u^\ell - x_v^\ell|$$

Consider some $i \neq \ell$

If $X_i = 1$ (i cuts u, v) $\Rightarrow B(s_\ell, r) \cap \{u, v\} \neq \emptyset$

$\Rightarrow S_i = 1$ only if i before ℓ in π

$$\Rightarrow Pr[S_i = 1 \mid X_i = 1] \le \frac{1}{2}$$

$$\Rightarrow \Pr[S_i = 1 \mid X_i = 1] \cdot \Pr[X_i = 1] \leq \frac{1}{2} |x_u^i - x_v^i|$$

Finish up:

$$\Pr[Z_{u,v} = 1] = \sum_{i=1}^{k} \Pr[S_i = 1 \mid X_i = 1] \cdot \Pr[X_i = 1]$$

$$\leq |x_u^\ell - x_v^\ell| + \sum_{i \neq \ell} \frac{1}{2} |x_u^i - x_v^i|$$

$$= \frac{1}{2} \underbrace{|x_u^\ell - x_v^\ell|}_{\leq \frac{1}{2}\|x_u - x_v\|_1} + \frac{1}{2} \|x_u - x_v\|_1$$

$$\leq \frac{3}{4} \|x_u - x_v\|_1$$

<u>Lemma</u>: $|x_u^\ell - x_v^\ell| \leq \frac{1}{2} \|x_u - x_v\|_1$

<u>Pf</u>: WLOG, $x_u^\ell \geq x_v^\ell$

$$\Rightarrow |x_u^\ell - x_v^\ell| = x_u^\ell - x_v^\ell = \Big(1 - \sum_{i \neq \ell} x_u^i\Big) - \Big(1 - \sum_{i \neq \ell} x_v^i\Big)$$

$$= \sum_{i \neq \ell} (x_v^i - x_u^i)$$

$$\leq \sum_{i \neq \ell} |x_v^i - x_u^i|$$

$$\Rightarrow 2|x_u^l - x_v^l| \leq \|x_u - x_v\|_1$$

$$\Rightarrow |x_u^l - x_v^l| \leq \frac{1}{2}\|x_u - x_v\|_1$$

Multicut:

Input: - $G = (V, E)$

- $c: E \to \mathbb{R}^+$

- Pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$

Feasible solution: $F \subseteq E$ s.t. $\forall i \in [k]$, no $s_i - t_i$ path in $G \setminus F$

Objective: min $c(F) = \sum_{e \in F} c(e)$

Thm: There is an $O(\log n)$-approximation for Multicut

Def: $P_i = \{s_i - t_i \text{ paths}\}$

LP:

$$\min \sum_{e \in E} c(e) x_e$$

$$\text{s.t.} \quad \sum_{e \in P} x_e \geq 1 \qquad \forall i \in [k], \forall P \in P_i$$

$$0 \leq x_e \leq 1 \qquad \forall e \in E$$

Solving LP: Ellipsoid, separation oracle = shortest path

Let x optimal LP solution

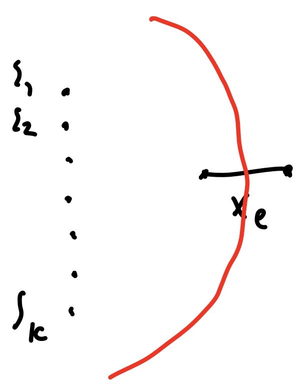
$$V^* = \sum_{e \in E} c(e)\, x_e = LP$$

$d: V \times V \to \mathbb{R}_{\geq 0}$ shortest path metric using x as edge lengths

$\Rightarrow d(s_i, t_i) \geq 1$ by LP (but no bound on $d(s_i, s_j)$)

How to round?

Ball around each s_i (like Multiway Cut);



Still want to use random-radius balls, but how?

Def: Given metric space (V, d), a
Low Diameter Random Decomposition (LDRD)
with parameter δ is a randomized algorithm
which creates partition $C_1, \ldots, C_\ell$ of V s.t.

1) $\text{diam}(C_i) \leq \delta \quad \forall\, i \in [\ell]$,

2) $\Pr[u, v \text{ in different clusters}] \leq O(\log n) \cdot \frac{d(u,v)}{\delta} \quad \forall\, u, v \in V$

Sps we have LDRD with parameter $1-\varepsilon$.

Apply to (V, d) to get $C_1, \ldots, C_\ell$, let $F = \bigcup_{i=1}^{\ell} \delta(C_i)$

$\Rightarrow$ since $d(s_i, t_i) \geq 1$, s_i, t_i in different clusters

$\Rightarrow$ s_i, t_i not connected in $G \setminus F$ $\Rightarrow$ Feasible

Let $Z_e = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$

$$E[c(F)] = E\left[\sum_{e \in E} c(e) Z_e\right] = \sum_{e \in E} c(e) E[Z_e]$$

$$\leq \sum_{e \in E} c(e) \cdot O(\log n) \frac{x_e}{1-\varepsilon}$$

$$\leq O(\log n) \sum_{e \in E} c(e) x_e$$

$$= O(\log n) \cdot V^* = O(\log n) \cdot LP$$

So just need to construct LDRD.

Already know how to do this!

One level of FRT!

- Choose r uniformly at random from $[\delta/4, \delta/2]$
- Choose a permutation π of V uniformly at random
- $S = V$
- for $j = 1$ to n {
  - $P = B(\pi(j), r) \cap S$
  - If $P \neq \emptyset$ {
    - Add P as a cluster
    - $S = S \setminus P$

    }

  }

Consider $u,v \in V$

$$S_w = \begin{cases} 1 & \text{if } w \text{ first vertex in } \pi \text{ s.t. } B(w,r) \cap \{u,v\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$X_w = \begin{cases} 1 & \text{if } |B(w,r) \cap \{u,v\}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

u,v in different clusters iff $\sum_{w \in V} S_w X_w = 1$

<u>Lemma</u>: For each $w \in V$ $\exists\, b_w \in \mathbb{R}$ s.t.

1) $Pr[S_w = 1 \mid X_w = 1] \leq b_w$, and

2) $\sum_{w \in V} b_w \leq O(\log n)$

<u>Pf</u>: See FRT

<u>Lemma</u>: $Pr[X_w = 1] \leq \frac{4d(u,v)}{\delta} \quad \forall w \in V$

<u>Pf</u>: WLOG, $d(w,u) \leq d(w,v)$

$$Pr[X_w = 1] = Pr[d(w,u) \leq r < d(w,v)]$$

$$\leq \frac{d(w,v) - d(w,u)}{\frac{\delta}{2} - \frac{\delta}{4}} \leq \frac{4d(u,v)}{\delta}$$

$$\Pr[u,v \text{ in different clusters}] = \Pr\left[\sum_{w \in V} S_w X_w = 1\right]$$

$$= E\left[\sum_{w \in V} S_w X_w\right] = \sum_{w \in V} E[S_w X_w]$$

$$= \sum_{w \in V} \Pr[S_w = 1 \wedge X_w = 1]$$

$$= \sum_{w \in V} \Pr[S_w = 1 \mid X_w = 1] \Pr[X_w = 1]$$

$$\leq \sum_{w \in V} b_w \cdot \frac{4d(u,v)}{\delta}$$

$$= \frac{4d(u,v)}{\delta} \sum_{w \in V} b_w$$

$$\leq O(\log n) \cdot \frac{d(u,v)}{\delta}$$