Max Coverage (Max k-cover):

Input: - Universe U, $|U| = n$
- Family of sets $S_1, S_2, \ldots, S_m$, each $S_i \subseteq U$
- Integer $k \leq n$

Feasible Solutions: $I \subseteq [m]$, $|I| \leq k$

Objective: Maximize $|\bigcup_{i \in I} S_i|$

Greedy Algorithm:

Same as for Set Cover, but stop after k iterations!

Thm: Greedy is a $(1 - \frac{1}{e})$-approximation

Pf: Like for Set Cover:

- g_t index of set picked by greedy in iteration t
- $I_t = \{g_1, g_2, \ldots, g_t\}$ the sets picked by greedy in first t iterations
- $J_t = U \setminus (\bigcup_{i \in I_t} S_i)$ the uncovered elements after t iterations

- $x_t = |S_{g_t} \cap J_{t-1}|$ # elements covered at iteration t
- OPT = # elements covered in optimal solution (and solution itself, abusing notation)

$-z_t = OPT - \sum_{i \le t} x_i = OPT - |\bigcup_{j \le t} S_{g_j}|$ $\quad$ not $|\bigcup_{i \in OPT} S_i \setminus \bigcup_{j \le t} S_j|$

Note: # elements covered by OPT, not covered by $\bigcup_{j \le t} S_{g_j}$, is at least z_t.

Claim: $x_{i+1} \ge \frac{z_i}{k}$

Pf: In iteration $i+1$, the k sets in OPT cover at least z_i uncovered elements

$\Rightarrow$ at least one of them covers $\ge \frac{z_i}{k}$ uncovered elements

$\Rightarrow$ greedy covers $\ge \frac{z_i}{k}$ uncovered elements

Claim: $z_i \le (1-\frac{1}{k})^i \cdot OPT$

Pf: induction on iteration i

$i=0$: $z_0 = OPT - 0 = OPT$ ✓

inductive step:

$$
\begin{aligned}
z_i &= z_{i-1} - x_i && \text{(def of } z_i\text{)} \\
&\le z_{i-1} - \frac{z_{i-1}}{k} && \text{(previous claim)} \\
&= (1-\tfrac{1}{k})\, z_{i-1} \\
&\le (1-\tfrac{1}{k})(1-\tfrac{1}{k})^{i-1} \cdot OPT && \text{(IH)} \\
&= (1-\tfrac{1}{k})^i \cdot OPT
\end{aligned}
$$

$$\text{Greedy} = \sum_{i=1}^{k} X_i = OPT - z_k$$

$$z_k = OPT - \sum_{i=1}^{k} x_i$$

$$\geq OPT - \left(1-\frac{1}{k}\right)^k \cdot OPT \quad \text{(previous claim)}$$

$$\geq OPT - \frac{1}{e} \cdot OPT \quad \left(\left(1-\frac{1}{k}\right)^k \leq \frac{1}{e}\right)$$

$$= \left(1-\frac{1}{e}\right) \cdot OPT$$

<u>Extensions</u>:

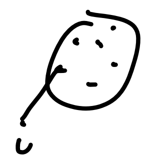
- Submodular optimization
- Minimum k-Union

$O(m^{1/4+\varepsilon})$-approx [CDM '17]

k-center:

Input: - (finite) metric (V, d), $|V| = n$

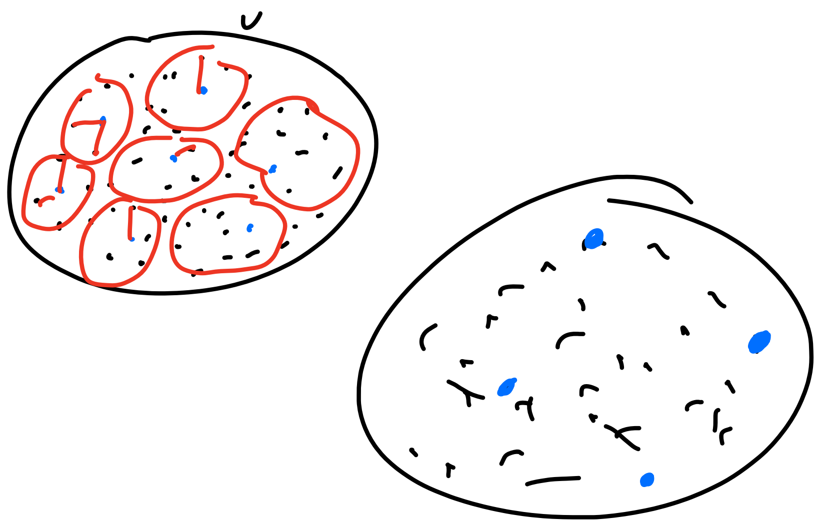
- integer k with $1 \le k \le n$



Feasible Solution: $F \subseteq V$ with $|F| = k$

Objective: minimize $\max_{v \in V} d(v, F)$

$$d(v, S) = \min_{u \in S} d(v, u)$$



Greedy:

Init: $F = \{u\}$ for some arbitrary $u \in V$

while $(|F| < k)$ {

Let $u \in V$ be node maximizing $d(u, F)$

Add u to F

}

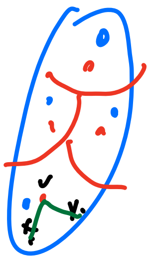
Thm: Greedy is a 2-approximation

Pf: Let F^* optimal solution, $OPT = \max_{u \in V} d(u, F^*)$

F solution returned by greedy

WTS: $\forall u \in V$, $d(u,F) \le 2 \cdot OPT = 2 \cdot \max_{v \in V} d(v, F^*)$

(not $d(u,F) \le 2 d(u, F^*)$)

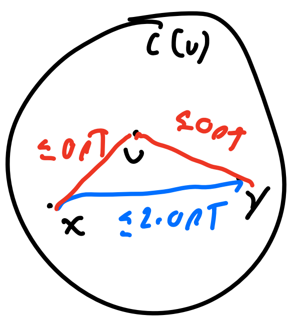


- For each $v \in F^*$, let cluster of v be

$$C(v) = \{u \in V : d(u,v) = d(u, F^*)\}$$

Lemma: Let $x, y \in C(v)$. Then $d(x,y) \le 2 \cdot OPT$

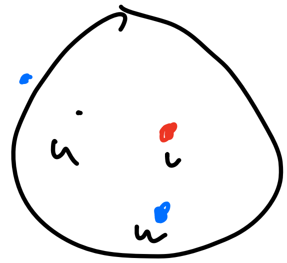
Pf:



$$\begin{aligned} d(x,y) &\le d(x,v) + d(v,y) \quad \Delta\text{-ineq} \\ &= d(x, F^*) + d(y, F^*) \\ &\le OPT + OPT = 2 \cdot OPT \end{aligned}$$

Consider arbitrary $u \in V$. WTS $d(u,F) \le 2 \cdot OPT$

Case 1: $\forall v \in F^*,\ C(v) \cap F \neq \emptyset$



Let $v \in F^*$ s.t. $u \in C(v)$.

$\Rightarrow \exists\ w \in C(v) \cap F$

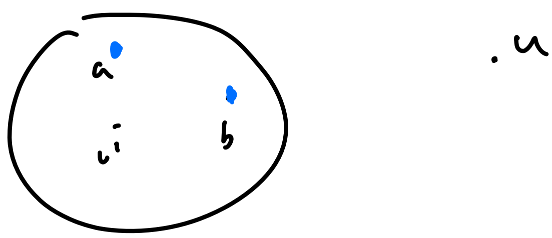
$\Rightarrow d(u, F) \leq d(u, w) \leq 2 \cdot OPT$

by def　　　by lemma

Case 2: $\exists\ v \in F^*$ s.t. $C(v) \cap F = \emptyset$

$\Rightarrow$ by pigeonhole, $\exists\ v' \in F^*$ s.t. $|C(v') \cap F| \geq 2$

Let $a, b \in C(v') \cap F$, with a added by greedy before b

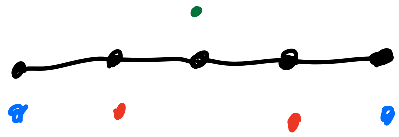


Let F' be nodes added by greedy until b added

$$
\begin{aligned}
d(u, F) &\leq d(u, F') && (F' \subset F) \\
&\leq d(b, F') && (\text{greedy alg}) \\
&\leq d(b, a) && (a \in F') \\
&\leq 2 \cdot OPT && (\text{by lemma})
\end{aligned}
$$

Q: Is analysis tight?

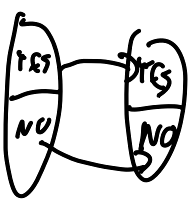
A: Yes!



Q: Is there a better algorithm?

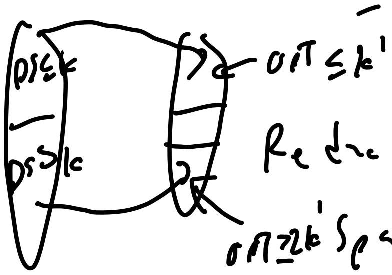
Thm: Assuming $P \neq NP$ there is no c-approx for k-center for any $c < 2$.

Pf: A dominating set in $G = (V, E)$ is a set $S \subseteq V$ s.t. every $v \in V$ is either in S or is adjacent to node in S



Dominating set problem: Given G, k, YES if G has a DS of size $\leq k$, NO otherwise

- NP-complete



Reduction: given $G = (V, E)$, k, create metric space (V, d) where

$$d(u,v) = \begin{cases} 1 & \text{if } \{u,v\} \in E \\ 2 & \text{otherwise} \end{cases}$$

<u>Lemma</u>: If G has a dominating set of size $\le k$, then (V,d) has a k-center solution of cost 1

<u>Pf</u>: Let $S \subseteq V$ be DS of G with $|S| \le k$

$\Rightarrow \forall u \in V, \exists\, s(u) \in S$ s.t. $\{u, s(u)\} \in E$

$\Rightarrow \forall u \in V, d(u,S) \le d(u, s(u)) = 1$

<u>Lemma</u>: If G <u>does not</u> have DS of size $\le k$, then OPT of k-center on $(V,d), k$ is ≥ 2

<u>Pf</u>: Contrapositive.

Sps (V,d) has k-center solution S of cost <2

$\Rightarrow \forall u, \; d(u,S) < 2 \Rightarrow d(u,S) = 1$ or 0

$\Rightarrow u \in S$ or adjacent to node in S

$\Rightarrow S$ a DS of size $\le k$

So sps had polytime $c<2$-approx for k-center.

Could solve Dominating Set:

- Given DS instance $(G=(V,E), k)$, create k-center instance
- Run c-approx alg, get back solution of cost α

If $\alpha < 2$, $\mathrm{OPT} \le 1 \Rightarrow$ YES of DS

If $\alpha \geq 2$, OPT > 1 ⇒ OPT ≥ 2 ⇒ NO of DS