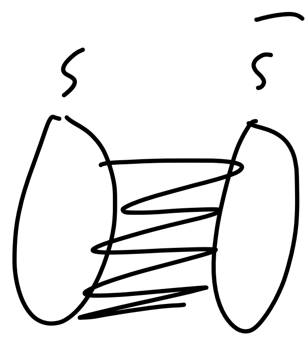
Local Search:

Start with feasible solution, make "local" "improving" changes until get to local optimum.

Main difficulties:

- running time
- showing local opt are close to global



$E(S,\bar{S})$

Max-Cut:

Input: $G=(V,E)$

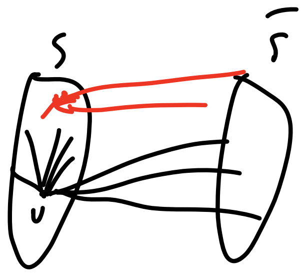
Feasible: $S \subseteq V$

Objective: max $|E(S,\bar{S})| = |\{\{u,v\} \in E : u \in S, v \in \bar{S}\}|$

NP-hard, unlike Min-Cut

Alg: - Start with arbitrary $S \subseteq V$

- While $\exists v \in V$ s.t. v has more edges to same side of cut than other side, move v to other side



Thm: Running time is polynomial

Pf:

Each iteration: poly time

iterations:

- Init: $|E(S, \bar{S})| \geq 0$
- $|E(S, \bar{S})|$ increases by ≥ 1 in every iteration
- $|E(S, \bar{S})| \leq m \leq n^2$

$\Rightarrow$ # iterations $\leq n^2$

Thm: Local Search is a 2-approximation

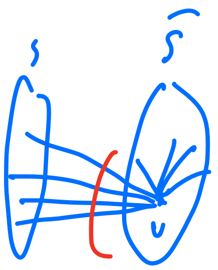
Pf: A local optimum is an $S \subseteq V$ where switching any vertex doesn't help.

Alg returns a local opt.

Claim: every local opt is a 2-approx

Let S be local opt,

- $d(v)$ degree of v,
- $d_{across}(v)$ # edges incident on v in $E(S, \bar{S})$



$d_{across}(v)$

$$|E(S,\bar{S})| = \frac{1}{2} \sum_{v \in V} d_{across}(v)$$

$$\geq \frac{1}{2} \sum_{v \in V} \frac{1}{2} d(v) \qquad (S \text{ local OPT})$$

$$= \frac{1}{4} \sum_{v \in V} d(v)$$

$$= \frac{1}{4} \cdot 2m = \frac{m}{2}$$

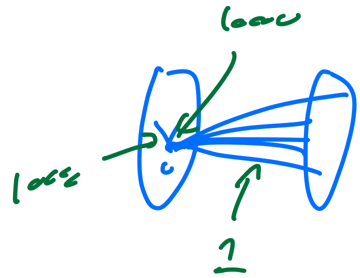
$\mathrm{OPT} \leq m$

$\Rightarrow$ 2-approx

Weighted Max-Cut:

Same as before, but also weights $w: E \to \mathbb{N}$

Goal: find $S \subseteq V$ maximizing $\sum_{e \in E(S,\bar{S})} w(e)$



Let $W = \sum_{e \in E} w(e)$

Previous algorithm:

switch v to other side if it increases weight across

Running time: - each iteration poly time

- $\leq W$ iterations

Input: n-ish bits for V — $\log W$ bits for each weight
m-ish bits for E

Need to change algorithm!

Notation:

- $\delta(v) = \{e \in E : v \in e\}$
- $w(E') = \sum_{e \in E'} w(e)$ $\qquad E' \subseteq E$
- $w(S) = w(E(S, \bar{S}))$ $\qquad S \subseteq V$

$= \sum_{e \in E(S,\bar{S})} w(e)$

New Local Search Alg:

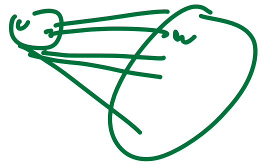
Init: $S = \{v\}$, where $v = \arg\max_{u \in V} w(\delta(u))$

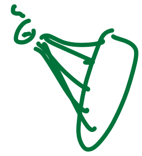
while $\begin{pmatrix} \exists v \in S \text{ s.t. } w(S \setminus \{v\}) \geq (1+\frac{\varepsilon}{n})\, w(S) \ \underline{\text{or}} \\ \exists v \in \bar{S} \text{ s.t. } w(S \cup \{v\}) \geq (1+\frac{\varepsilon}{n})\, w(S) \end{pmatrix}$

change side of v

<u>Thm</u>: # iterations $\leq O(\frac{1}{\varepsilon} n \log n)$

<u>Pf</u>: Let $S_0 = \{v\}, S_1, S_2, \ldots, S_k$ be sets created by the algorithm.





$$\Rightarrow \quad w(S_0) \geq \frac{2W}{n}$$

$$\Rightarrow w(S_i) \geq \left(1+\frac{\varepsilon}{n}\right) w(S_{i-1}) \geq \left(1+\frac{\varepsilon}{n}\right)^2 w(S_{i-2}) \geq \dots$$

$$\geq \left(1+\frac{\varepsilon}{n}\right)^i w(S_0)$$

$$= \left(1+\frac{\varepsilon}{n}\right)^i \frac{W}{n}$$

$$\Rightarrow w(S_k) \geq \left(1+\frac{\varepsilon}{n}\right)^k \frac{W}{n}$$

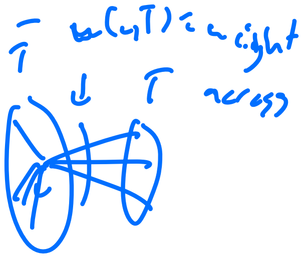
$$\geq \left(1+\frac{\varepsilon}{n}\right)^k \frac{w(S_k)}{n}$$

$$\Rightarrow \left(1+\frac{\varepsilon}{n}\right)^k \leq n$$

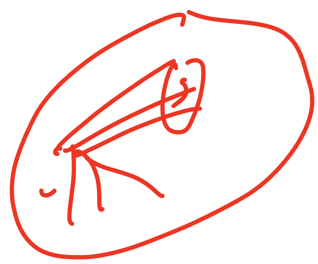
$$\Rightarrow k \log\left(1+\frac{\varepsilon}{n}\right) \leq \log n$$

$$\Rightarrow k \leq \frac{\log n}{\log\left(1+\frac{\varepsilon}{n}\right)} \leq \frac{\log n}{\frac{\varepsilon}{2n}}$$

$\log(1+x) \geq \frac{x}{2}$ for $0 \leq x \leq 5$



$$= O\left(\frac{1}{\varepsilon} n \log n\right)$$



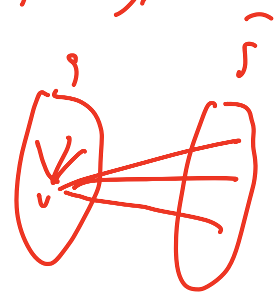
Thm: Alg is a $(2+\varepsilon)$-approx

Pf: Notation: $w(v,T) = \sum_{u \in T : \{v,u\} \in E} w(\{v,u\})$

Let S be set returned by algorithm.

Then $\forall v \in S$:

$$w(S \setminus \{v\}) \leq (1 + \tfrac{\varepsilon}{n}) w(S) \quad \text{(def of alg)}$$



$$\Rightarrow w(S \setminus \{v\}) - w(S) \leq \tfrac{\varepsilon}{n} w(S)$$

$$\Rightarrow w(v, S) - w(v, \bar{S}) \leq \tfrac{\varepsilon}{n} w(S)$$

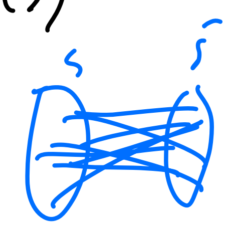
$$\Rightarrow w(v, \bar{S}) \geq w(v, S) - \tfrac{\varepsilon}{n} w(S)$$

$$\Rightarrow 2w(v, \bar{S}) \geq w(\delta(v)) - \tfrac{\varepsilon}{n} w(S) \quad \text{(add } w(v, \bar{S}) \text{ to both sides)}$$

$$\Rightarrow w(v, \bar{S}) \geq \tfrac{1}{2} w(\delta(v)) - \tfrac{\varepsilon}{2n} w(S)$$

Symmetric argument:

$$\forall v \in \bar{S}, \quad w(v, S) \geq \tfrac{1}{2} w(\delta(v)) - \tfrac{\varepsilon}{2n} w(S)$$



$$\Rightarrow 2w(S) = \sum_{v \in S} w(v, \bar{S}) + \sum_{v \in \bar{S}} w(v, S)$$

$$\geq \sum_{v \in S} \left(\tfrac{1}{2} w(\delta(v)) - \tfrac{\varepsilon}{2n} w(S) \right) + \sum_{v \in \bar{S}} \left(\tfrac{1}{2} w(\delta(v)) - \tfrac{\varepsilon}{2n} w(S) \right)$$

$$= \sum_{v \in V} \left(\tfrac{1}{2} w(\delta(v)) - \tfrac{\varepsilon}{2n} w(S) \right)$$

$$= W - \frac{\varepsilon}{2} w(S)$$

$$\Rightarrow \left(2 + \frac{\varepsilon}{2}\right) w(S) \geq W$$

$$\Rightarrow \quad w(S) \geq \frac{W}{2 + \frac{\varepsilon}{2}} \geq \frac{OPT}{2 + \frac{\varepsilon}{2}}$$