Notation: - $d_G(v)$ = degree of v in G

- $\Delta(G) = \max_{v \in V} d_G(v)$ = max degree

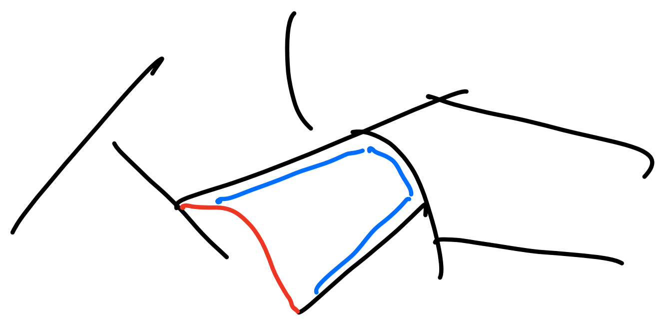
Minimum Degree Spanning Tree:

Input: $G = (V, E)$

Feasible solution: Spanning tree T of G

Objective: min $\Delta(T)$

NP-hard (see homework)



Def: For spanning tree T, non-tree edge e, the fundamental cycle of e is the unique cycle in $T \cup \{e\}$

Local move:

pair (e, e'), $e \notin T$, e' in fundamental cycle of e.

Add e, remove e'

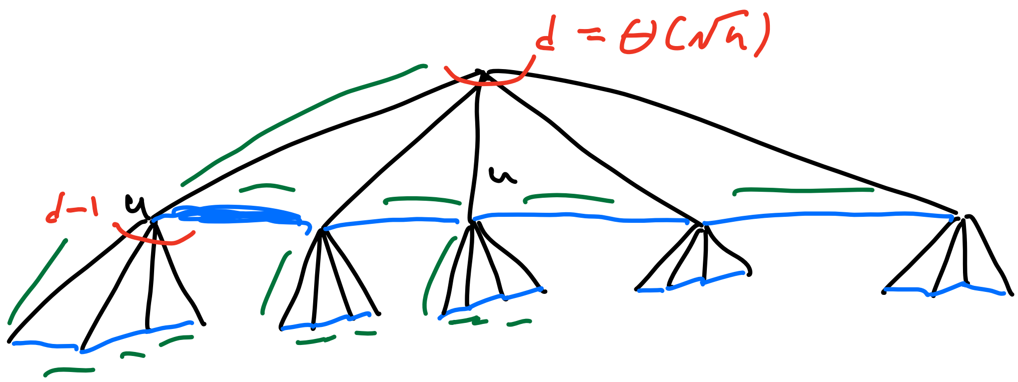
Local Search algorithm:

Init T an arbitrary spanning tree

while there is a local move which decreases $\Delta(T)$

do it

Problem: might be bad approximation!



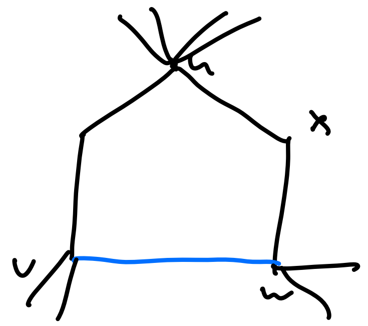
Local opt: max degree $d = \Theta(\sqrt{n})$

Global opt: max degree 3

Def: Let $u \in V$. $(u, \{v,w\})$ is a u-improvement if $\exists$ $\{u,x\}$ on fundamental cycle of $\{v,w\}$, and if we add $\{v,w\}$ and remove $\{u,x\}$ to get T' then

$$\max(d_{T'}(v), d_{T'}(w)) \leq d_{T'}(u) = d_T(u) - 1$$

Example:



LS Alg 2:

while $\exists$ u-improvement for some node u, do it

Problem: running time!

LS Alg 3:

while $\exists$ u-improvement where $d_T(u) \geq \Delta(T) - \log n$

do it.

Thm: Running time is polynomial

Pf:

Each iteration polytime $\Rightarrow$ just need to bound # iterations

Potential function:

$$\Phi(v) = 3^{d_T(v)} \qquad \Phi(T) = \sum_{v \in V} \Phi(v) = \sum_{v \in V} 3^{d_T(v)}$$

Easy facts:

- $\Phi(T) \geq 3n$
- $\Phi(T) \leq n \cdot 3^n$

Claim: Suppose did a u-improvement on T to get T', where $d_T(u) \geq \Delta(T) - \log n$. Then

$$\Phi(T') \leq \left(1 - \frac{2}{9n^3}\right) \Phi(T)$$

Pf of claim:

Let $(u, \{v,w\})$ be the u-improvement, and

let $i = d_T(u) \geq \Delta(T) - \log n$

Decrease in Φ $(\Phi(T) - \Phi(T'))$

$u: \Phi_T(u) - \Phi_{T'}(u) = 3^i - 3^{i-1} = 2 \cdot 3^{i-1}$

$x: \geq 0$

$v: 3^{d_T(v)} - 3^{d_{T'}(v)} = -(3^{d_{T'}(v)} - 3^{d_T(v)})$

$= -(3^{d_{T'}(v)} - 3^{d_{T'}(v)-1})$

$= -3^{d_{T'}(v)-1}(3-1)$

$= -2 \cdot 3^{d_{T'}(v)-1}$ $\quad d_{T'}(v) \leq d_{T'}(u)$

$\geq -2 \cdot 3^{d_{T'}(u)-1}$

$= -2 \cdot 3^{i-2}$

$w:$ same

All other nodes: unchanged

$\Rightarrow \Phi(T) - \Phi(T') \geq 2 \cdot 3^{i-1} - 4 \cdot 3^{i-2}$

$= \frac{2}{3} \cdot 3^i - \frac{4}{9} \cdot 3^i$

$= \frac{2}{9} \cdot 3^i$

$$\geq \frac{2}{9} \cdot 3^{\Delta(T) - \log n}$$

$$= \frac{2}{9 \cdot 3^{\log n}} \cdot 3^{\Delta(T)}$$

$$= \frac{2}{9 \cdot n^{\log 3}} \cdot 3^{\Delta(T)}$$

$$\geq \frac{2}{9 n^2} \cdot \frac{1}{n} \Phi(T)$$

$$= \frac{2}{9 n^3} \Phi(T)$$

$$\Rightarrow \Phi(T') \leq \left(1 - \frac{2}{9n^3}\right) \Phi(T)$$ ✓

Use claim: suppose runs for $\frac{9}{2} n^4 \ln 3$ iterations

$$\Rightarrow \Phi(T) \leq \left(1 - \frac{2}{9n^3}\right)^{\frac{9}{2} n^4 \ln 3} \cdot n \cdot 3^n$$

$$\leq e^{-n \ln 3} \cdot n 3^n$$

$$= n$$

Since $\Phi(T) \geq 3n$, must have already finished ✓

Approximation:

Let T be tree returned by alg,

T^* optimal tree, $\Delta^* = \Delta(T^*)$

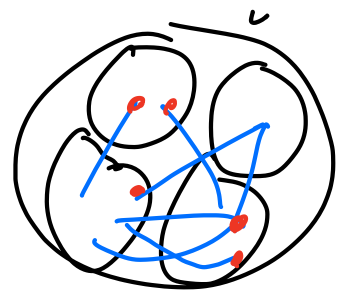
Thm: $\Delta(T) \leq 2 \cdot \Delta^* + \log n + 1$

Lemma: Let $(V_1, V_2, \dots, V_k)$ be a partition of V,

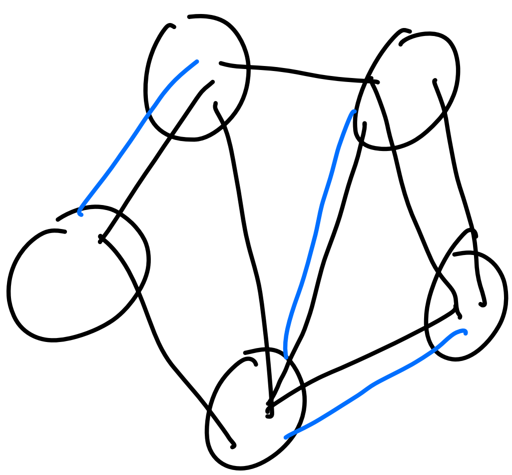
let $E' \subseteq E$ edges across partition,

let V' a vertex cover of E'.

Then $\Delta^* \geq \frac{k-1}{|V'|}$



Pf:



T^* has $\geq k-1$ edges of E'

Each of which has ≥ 1 endpoint in V'

$\Rightarrow$ some vertex $v \in V'$ has

$d_{T^*}(v) \geq \frac{k-1}{|V'|}$

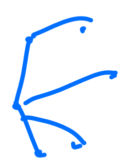
From now on: $i \geq \Delta(T) - \log n$

Def: Let $S_i = \{v \in V : d_T(v) \geq i\}$

Def: Let E_i^T be edges of T incident on at least one node in S_i

Claim: $|E_i^T| \geq (i-1)|S_i| + 1$

Pf: $\sum_{v \in S_i} d_T(v) \geq i|S_i|$ (def of S_i)

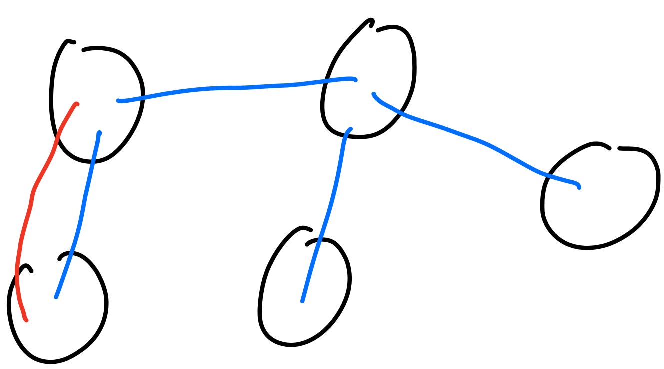


$\left|\{\{u,v\} \in E_i^T : u,v \in S_i\}\right| \leq |S_i| - 1$ (T a tree)

$\Rightarrow |E_i^T| = \sum_{v \in S_i} d_T(v) - \left|\{\{u,v\} \in E_i^T : u,v \in S_i\}\right|$

$\geq i|S_i| - (|S_i| - 1) = (i-1)|S_i| + 1$

Def: Let E_i^G be edges in G between components of $T - E_i^T$



(partition we'll apply first lemma to)

<u>Claim</u>: S_{i-1} is a vertex cover of E_i^G

<u>Pf</u>: Let $e = \{x, y\} \in E_i^G$

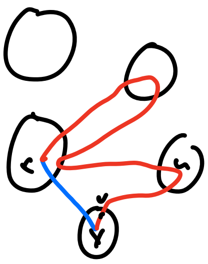
Case 1: $e \in E_i^T$.

By def of E_i^T, either x or y (or both) is in $S_i \Rightarrow$ in S_{i-1} ✓

Case 2: $e \notin E_i^T$

$\Rightarrow e \notin T$

$\Rightarrow$ Since $e \notin T$, has fundamental cycle C



$\Rightarrow$ since x, y in diff components of $T - E_i^T$, C has some edge $\{u, v\}$ from E_i^T

$\Rightarrow$ either u or v in S_i : WLOG, $u \in S_i$

$\Rightarrow$ since at local opt, $(u, \{x, y\})$ <u>cannot</u> be u-improvement

$\Rightarrow \max(d_T(x)+1, d_T(y)+1) \geq d_T(u) \geq i$

$\Rightarrow$ either x or y in S_{i-1}

Claim: $\exists\, i \geq \Delta(T) - \log n$ s.t. $|S_{i-1}| \leq 2|S_i|$

Pf:

Sps false $\Rightarrow |S_{i-1}| > 2|S_i| \quad \forall i$

$|S_{\Delta(T)}| \geq 1 \Rightarrow |S_{\Delta(T)-\log n}| > 2^{\log n} \cdot 1 = n$

$\Rightarrow\Leftarrow$

Can finally apply partition lemma!

Let $i^* \geq \Delta(T) - \log n$ s.t. $|S_{i^*-1}| \leq 2|S_{i^*}|$

Partition V into $|E^T_{i^*}| + 1$ sets by $T - E^T_{i^*}$

S_{i^*-1} a vertex cover of $E^G_{i^*}$

$$\Rightarrow \Delta^* \geq \frac{|E^T_{i^*}|}{|S_{i^*-1}|}$$

$$\geq \frac{(i^*-1)|S_{i^*}| + 1}{2|S_{i^*}|}$$

$$\geq \frac{i^*-1}{2}$$

$$\geq \frac{\Delta(T) - \log n - 1}{2}$$

$$\Rightarrow \Delta(T) \leq 2\Delta^* + \log n + 1$$

Better: - $\Delta(T) \leq \Delta^* + 1$

- Min-degree Steiner Tree

Distributed [D-H-I-N '19]: $\Delta(T) \leq O(\Delta^* + \log n)$