LPs for Approximation Algorithms:

To prove α-approx, often done:

1) Prove $OPT \geq LB$
2) Prove $ALG \leq \alpha \cdot LB$

$\Rightarrow ALG \leq \alpha \cdot OPT$

TSP: LB = MST

Vertex Cover: LB = max matching

Steiner Tree: LB = MST of terminals

Linear Programming: automatically generate a LB, which can be modified algorithmically!

Example: Weighted Vertex Cover

Input: $-G = (V, E)$
$- c: V \to \mathbb{R}^+$

Feasible Solution: $S \subseteq V$ s.t. $S \cap \{u,v\} \neq \emptyset \quad \forall \{u,v\} \in E$

Objective: $\min c(S) = \sum_{v \in S} c(v)$

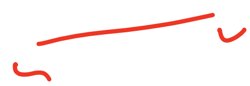
Integer Linear Programming:

- Variables $x_1, \dots, x_n$, each of which must be an integer
- m linear inequalities over variables $\quad a^T x \le b$ $\quad \sum_{i=1}^{n} a_i x_i \le b$
- (Possibly) linear objective

ILP for Weighted Vertex Cover:

Vars: $x_v \quad \forall v \in V$

$$\min \sum_{v \in V} c(v) x_v$$



$$\text{s.t.} \quad x_u + x_v \ge 1 \qquad \forall \{u,v\} \in E$$

$$x_v \in \{0,1\} \qquad \forall v \in V$$

$$x_v \ge 0 \qquad \forall v \in V$$
$$x_v \le 1$$

Thm: This ILP is an exact formulation of WVC

Pf:

Let S be a VC. Set $x_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \notin S \end{cases}$

Let $\{u,v\} \in E \Rightarrow x_u + x_v \ge 1$ (S a VC)

$\Rightarrow x$ a feasible ILP solution

$$\sum_{v \in V} c(v) x_v = \sum_{v \in S} c(v) = c(S)$$

$\Rightarrow OPT(ILP) \le OPT(WVC)$

Let x be an ILP solution

Let $S = \{v \in V : x_v = 1\}$

Let $\{u,v\} \in E \Rightarrow \{u,v\} \cap S \neq \emptyset$ (since $x_u + x_v \geq 1$)

$\Rightarrow S$ a VC

$$c(S) = \sum_{v \in S} c(v) = \sum_{v \in V} c(v) x_v$$

$\Rightarrow OPT(WVC) \leq OPT(ILP)$

So ILP exactly the same ($\Rightarrow$ NP-hard)

why did we do this?

<u>Linear Program</u>:

Same thing, no integrality constraints; variables take values in $\mathbb{R}$ (really $\mathbb{Q}$)

Polytime solvable!

"Relax" ILP to an LP

$$\min \sum_{v \in V} c(v)\, x_v$$

$$\text{s.t.} \quad x_u + x_v \geq 1 \qquad \forall \{u,v\} \in E$$

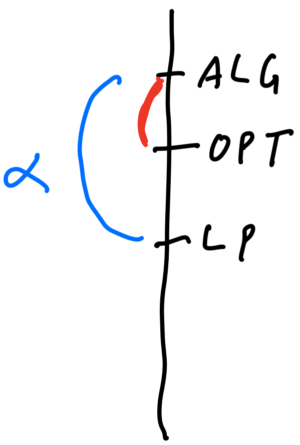
$$0 \leq x_v \leq 1 \qquad \forall v \in V$$

Can solve this!

Key point: Every ILP solution x also an LP solution!

(Including ILP opt x^*_{ILP})

$$\Rightarrow OPT(LP) = c(x^*_{LP}) \leq c(x^*_{ILP}) = OPT(ILP) = OPT(WVC)$$



If we can find a vertex cover (equivalently ILP solution) of cost $\leq \alpha \cdot LP$, also $\leq \alpha \cdot OPT$

Algorithmic idea: LP rounding

1) Write exact ILP formulation

2) Relax to LP, so $OPT(LP) \le OPT(ILP)$

3) Solve LP relaxation optimally, get solution x^*

4) "Round" x^* to integer values to get an ILP solution

(try to lose small α in rounding)

<u>LP Rounding for WVC:</u>

$$\min \sum_{v \in V} c(v)\, x_v$$

$$\text{s.t.} \quad x_u + x_v \ge 1 \qquad \forall \{u,v\} \in E$$

$$0 \le x_v \le 1 \qquad \forall v \in V$$

Solve to get x^*. Want integral solution x'

$$x'_v = \begin{cases} 1 & \text{if } x^*_v \ge \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

<u>Thm</u>: x' is a feasible ILP solution

Pf:

Clearly $x'_v \in \{0,1\} \quad \forall v \in V$

Let $\{u,v\} \in E$

$\Rightarrow x^*_u + x^*_v \geq 1$ since x^* feasible for LP

$\Rightarrow \max(x^*_u, x^*_v) \geq \frac{1}{2}$

$\Rightarrow x'_u + x'_v \geq 1$

Thm: $c(x') \leq 2 \cdot c(x^*)$

Pf:

$$
\begin{aligned}
c(x') &= \sum_{v \in V} c(v)\, x'_v \\
&= \sum_{v : x^*_v \geq \frac{1}{2}} c(v)\, 1 \\
&\leq \sum_{v \in V} c(v) \cdot 2 x^*_v \\
&= 2 \cdot c(x^*)
\end{aligned}
$$

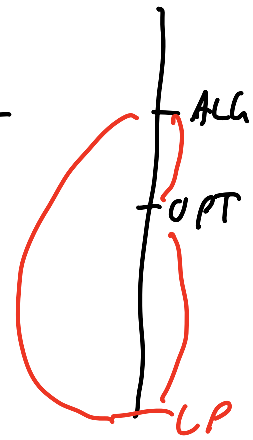
$\Rightarrow$ 2-approximation!

Integrality gaps:

Key idea of approach:

1) $LP \le OPT$
2) $ALG \le \alpha \cdot LP$

$\Rightarrow \quad ALG \le \alpha \cdot OPT$



Hopeless if $LP \ll OPT$!

$\Rightarrow \frac{OPT}{LP}$ is the best approximation we can hope for from this approach

Def: The integrality gap of an LP relaxation for a (minimization) problem Π is

$$\max_{\text{instances } I \text{ of } \Pi} \left(\frac{OPT(I)}{LP(I)} \right)$$

integral opt

fractional / LP opt

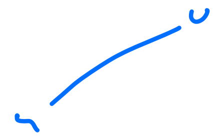
Integrality gap for WVC:

Thm: The integrality gap for WVC LP is $\ge 2(1 - \frac{1}{n})$

Pf: Let $G = K_n$, $c(v) = 1 \;\; \forall v \in V$

$\Rightarrow OPT = n-1$

LP: Set $x_v = \frac{1}{2} \;\; \forall v \in V$



Feasible LP solution

$\Rightarrow LP \le \frac{1}{2} \cdot n$

$$\Rightarrow \frac{OPT}{LP} \ge \frac{n-1}{\frac{1}{2} \cdot n} = 2\left(1 - \frac{1}{n}\right)$$

Max Independent Set:

Input: $G = (V, E)$

Feasible solution: $S \subseteq V$ s.t. $|e \cap S| \le 1 \;\; \forall e \in E$

Objective: max $|S|$

$$\max \sum_{v \in V} x_v$$

$$\text{s.t.} \quad x_u + x_v \le 1 \qquad \forall \{u,v\} \in E$$

$$0 \le x_v \le 1 \qquad \forall v \in V$$

Thm: Integrality gap $\geq \frac{n}{2}$

Pf: $G = K_n$

$\Rightarrow OPT = 1$

LP: Set $x_v = \frac{1}{2} \ \forall v \in V$

$\Rightarrow$ feasible solution

$\Rightarrow LP \geq \frac{1}{2} \cdot n$

$$\Rightarrow \frac{LP}{OPT} \geq \frac{\frac{1}{2} \cdot n}{1} = \frac{n}{2}$$

Solving LPs:

General LP:

$$\begin{aligned} \min \ & c^T x & & c \in \mathbb{Q}^n \\ \text{s.t. } & Ax \geq b & & A \in \mathbb{Q}^{m \times n} \\ & x \geq 0 & & b \in \mathbb{Q}^m \end{aligned}$$

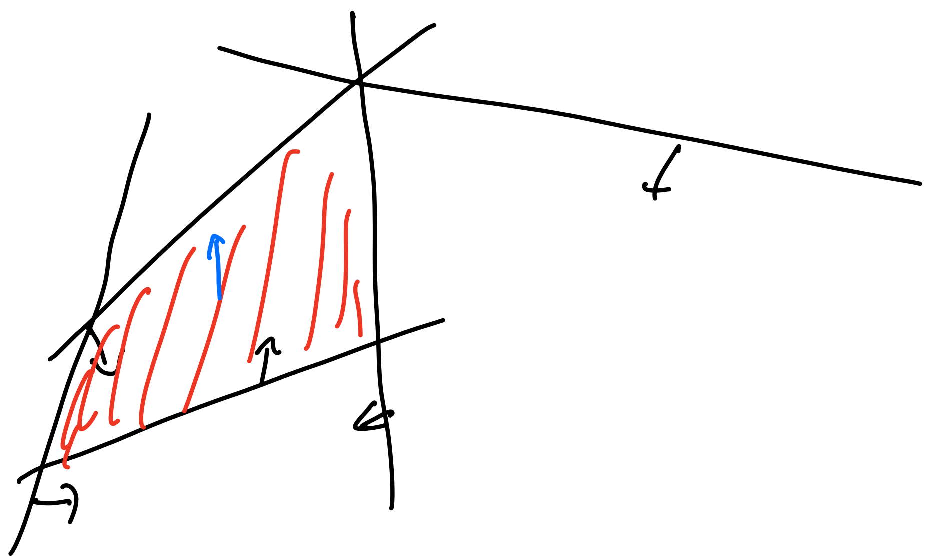
Let Δ be bit-complexity of an LP: # bits needed to write any coefficient (a_{ij}, c_i, b_j)

Thm: Linear Programming can be solved in time
$\text{poly}(n, m, \Delta)$

Intuition: think geometrically!

LP constraints $\rightarrow$ polytope in $\mathbb{R}^n$ with

Objective: direction to optimize



Simplex: local search on vertices of polytope

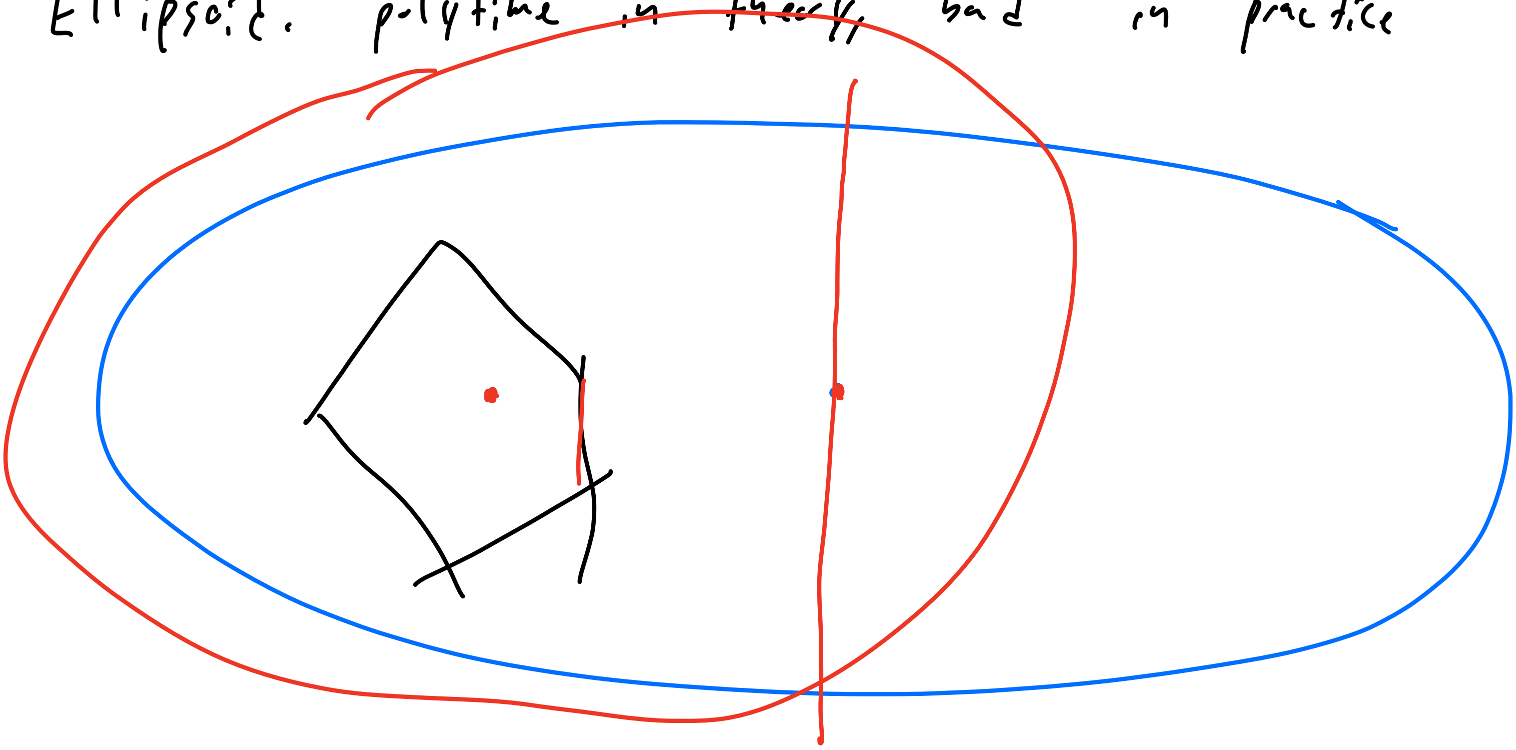
Good in practice, not polynomial time in worst case!

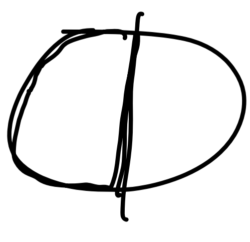
Interior Point methods:

Complicated algorithms that walk inside polytope

Good in practice, polytime in worst case!

Ellipsoid: polytime in theory, bad in practice





Key fact: just need to be able to separate!

- Given x,
  - if x in polytope return yes
  - if x not in polytope, find separating hyperplane
    (violated constraint)

Can solve LPs with exponential # constraints if can separate!

Ex: Spanning tree polytope

$$\min \sum_{e \in E} c(e) x_e$$

$$\text{s.t.} \sum_{e \in E(S, \bar{S})} x_e \geq 1 \qquad \forall S \subseteq V,\ S \neq \emptyset, V$$

$$0 \leq x_e \leq 1$$

Exponential constraints!

Separation: given x, is there a violated constraint?

$\Rightarrow$ Is there $S \subseteq V$ s.t. $\sum_{e \in E(S,\bar{S})} x_e < 1$?

Compute min cut!