Lecture 10: Universal and Perfect Hashing

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601.433/633 Introduction to Algorithms

Introduction

Another approach to dictionaries (insert, lookup, delete): hashing

- Can improve operations to $O(1)$, but with many caveats!

Should have seen some discussion of hashing in data structures. Also in CLRS.

- Separate chaining vs. open addressing

Today: discussion of caveats, more advanced versions of hashing (universal and perfect)

Hashing Basics

- Keys from universe U (think very large)
- Set $S \subseteq U$ of keys we actually care about (think relatively small). $|S| = N$.
- Hash table A (array) of size M.
- Hash function $h : U \to [M]$
  - $[M] = \{1, 2, \ldots, M\}$
- Idea: store x in $A[h(x)]$

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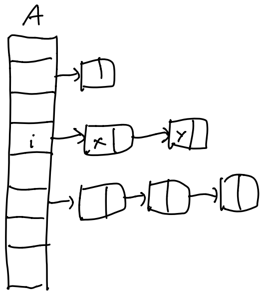
One more component: *collision resolution*

- Today: *separate chaining*
- $A[i]$ is a linked list containing all x inserted where $h(x) = i$.



Dictionary Operations

Lookup(x): Walk down the list at $A[h(x)]$ until we find x (or walk to the end of the list)

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- Small M. Ideally, $M = O(N)$.
- h fast to compute.

Bad News

Theorem

For any hash function h, if $|U| \geq (N-1)M + 1$, then there exists a set S of N elements that all hash to the same location.

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- Option 1: don't worry about it, hope world isn't adversarial.
- Option 2: Randomness! *Random function* $h : U \to [M]$
  - For each $x \in U$, choose $y \in [M]$ uniformly at random and set $h(x) = y$.
  - Hopefully good behavior in expectation.

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- Option 2: Randomness! *Random function* $h : U \to [M]$
  - For each $x \in U$, choose $y \in [M]$ uniformly at random and set $h(x) = y$.
  - Hopefully good behavior in expectation.
  - Problem: How can we store/remember/create h?

Universal Hashing

Definition

A probability distribution H over hash functions $\{h : U \to [M]\}$ is *universal* if

$$\Pr_{h \sim H}[h(x) = h(y)] \leq 1/M$$

for all $x, y \in U$ with $x \neq y$.

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So Lookup(x) and Delete(x) have expected time $O(N/M)$.
$\implies$ If $M = \Omega(N)$, operations in $O(1)$ time!

Main Proof

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Proof.

Let $C_{xy} = \begin{cases} 1 & \text{if } h(x) = h(y) \\ 0 & \text{otherwise} \end{cases}$

$$\implies E[C_{xy}] = \Pr_{h \sim H}[h(x) = h(y)] \leq 1/M$$

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Number of collisions between x and S is exactly $\sum_{y \in S} C_{xy}$

$$\implies E\left[\sum_{y \in S} C_{xy}\right] = \sum_{y \in S} E[C_{xy}] \leq \sum_{y \in S} \frac{1}{M} = N/M$$

□

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So universal distributions are great. Can we construct them?

Universal Hash Families

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Often use H to refer to both set of functions and uniform distribution over it.

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Notation:

- $U = \{0,1\}^u$ (so $|U| = 2^u$)
- $M = 2^b$, so an index to A is an element of $\{0,1\}^b$

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Construction: $H = \{0,1\}^{b \times u}$, i.e., H is all $b \times u$ binary matrices

- Each $h \in H$ is a (linear) function from U to $[M]$: $h(x) = hx \in \{0,1\}^b$ (all operations mod 2)

$$\underset{h}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}} \underset{x}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}} = \underset{h(x)}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

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- $h(x) = h'(x)$ already fixed.
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- Happens with probability exactly $1/2^b = 1/M$ □

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$$\Pr_{h\sim H}[\exists \text{ collision in } S] \leq \sum_{\substack{x,y\in S\\ x\neq y}} \Pr_{h\sim H}[h(x) = h(y)] \leq \sum_{\substack{x,y\in S\\ x\neq y}} \frac{1}{N^2}$$

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So keep sampling $h \sim H$ until get one with no collisions!

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- Use Method 1: $O(n_i^2)$-size perfect hashing of S_i.
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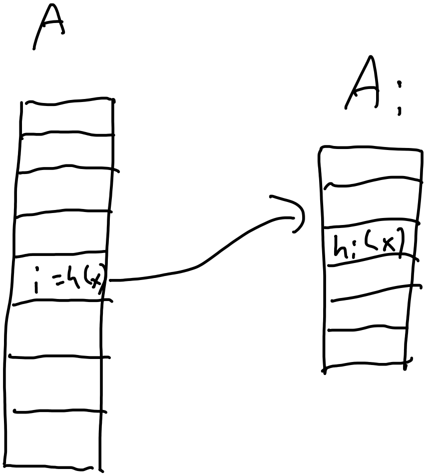
Lookup(x): Look in $A_{h(x)}[h_{h(x)}(x)]$

Picture



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Lookup time: by analysis of Method 1, no collisions in second level.

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Theorem

Let H be universal onto a table of size N. Then

$$\Pr_{h \sim H}\left[\sum_{i=1}^{N} n_i^2 > 4N\right] < 1/2.$$

So like with method 1: keep drawing $h \sim H$ until $\sum_{i=1}^{N} n_i^2 \leq 4N$

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Prove that $E\left[\sum_{i=1}^{N} n_i^2\right] \le 2N$.

- Implies theorem by Markov's inequality
  - $\Pr[X > 2E[X]] \le 1/2$ for nonnegative random variables X.

Proof

Observation: $\sum_{i=1}^{N} n_i^2$ is exactly number of *ordered* pairs that collide, including self-collisions

- Example: If $S_i = \{a, b, c\}$ then $n_i^2 = 9$. Ordered colliding pairs: $(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)$

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Let $C_{xy} = \begin{cases} 1 & \text{if } h(x) = h(y) \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned}
E\left[\sum_{i=1}^{N} n_i^2\right] &= E\left[\sum_{x\in S}\sum_{y\in S} C_{xy}\right] \\
&= N + \sum_{x\in S}\sum_{y\in S: y\neq x} E\left[C_{xy}\right] && \text{(linearity of expectations)} \\
&\leq N + \frac{N(N-1)}{M} && \text{(definition of universal)} \\
&< 2N && \text{(since } M = N\text{)}
\end{aligned}$$