# Lecture 12: Dynamic Programming II

Michael Dinitz

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601.433/633 Introduction to Algorithms

# Introduction

Today: two more examples of dynamic programming

- *Longest Common Subsequence* (strings)
- *Optimal Binary Search Tree* (trees)

Important problems, but really: more examples of dynamic programming

Both in CLRS (unlike Weighted Interval Scheduling)

# Longest Common Subsequence

# Definitions

**String:** Sequence of elements of some *alphabet* ($\{0, 1\}$, or $\{A - Z\} \cup \{a - z\}$, etc.)

> **Definition:** A sequence $Z = (z_1, \dots, z_k)$ is a *subsequence* of $X = (x_1, \dots, x_m)$ if there exists a strictly increasing sequence $(i_1, i_2, \dots, i_k)$ such that $x_{i_j} = z_j$ for all $j \in \{1, 2, \dots, k\}$.

**Example:** $(B, C, D, B)$ is a subsequence of $(A, B, C, B, D, A, B)$

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- Allowed to skip positions, unlike substring!

**Definition:** In *Longest Common Subsequence* problem (LCS) we are given two strings $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots y_n)$. Need to find the longest $Z$ which is a subsequence of both $X$ and $Y$.

# Subproblems

First and most important step of dynamic programming: define subproblems!

- Not obvious: $\boldsymbol{X}$ and $\boldsymbol{Y}$ might not even be same length!

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Prefixes of strings

- $\boldsymbol{X_i = (x_1, x_2, \ldots, x_i)}$ (so $\boldsymbol{X = X_m}$)
- $\boldsymbol{Y_j = (y_1, y_2, \ldots, y_j)}$ (so $\boldsymbol{Y = Y_n}$)

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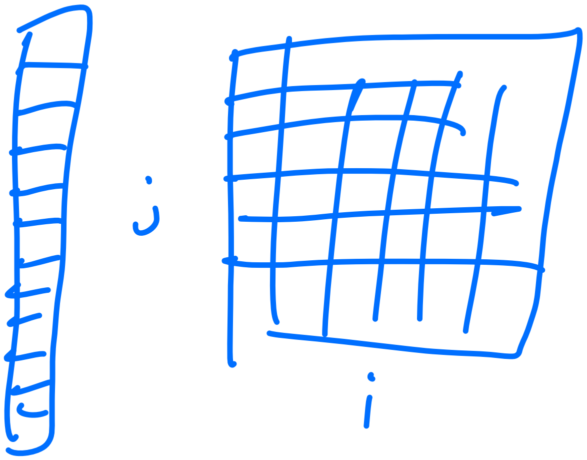


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**Definition:** Let $OPT(i, j)$ be longest common subsequence of $X_i$ and $Y_j$

So looking for optimal solution $OPT = OPT(m, n)$

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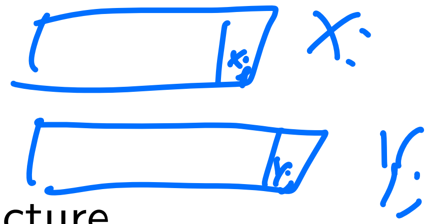
Two-dimensional table!

# Optimal Substructure

Second step of dynamic programming: prove optimal substructure

- Relationship between subproblems: show that solution to subproblem can be found from solutions to smaller subproblems

# Optimal Substructure

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## Theorem

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# Optimal Substructure: Proof (I)

**Case 1:** If $x_i = y_j$, then $z_k = x_i = y_j$ and $Z_{k-1} = OPT(i-1, j-1)$

## Proof Sketch.

Contradiction.

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**Part 2**: Suppose $Z_{k-1} \neq OPT(i-1, j-1)$.

$\implies$ $\exists W$ LCS of $X_{i-1}, Y_{j-1}$ of length $> k-1 \implies \geq k$

$\implies$ $(W, a)$ common subsequence of $X_i, Y_j$ of length $> k$

- Contradiction to $Z$ being LCS of $X_i$ and $Y_j$ $\square$

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$\implies |OPT(i-1,j)| \leq |OPT(i,j)| = |Z| \qquad$ (def of $OPT(i,j)$ and $Z$)

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**Proof.**

Since $z_k \neq x_i$, $Z$ a common subsequence of $X_{i-1}, Y_j$ $\Rightarrow OPT(i-1, j) \geq |Z|$

$OPT(i-1, j)$ a common subsequence of $X_i, Y_j$

$\implies |OPT(i-1,j)| \leq |OPT(i,j)| = |Z|$ (def of $OPT(i,j)$ and $Z$)

$\implies Z = OPT(i-1, j)$ □

# Optimal Substructure: Proof (III)

**Case 3:** If $x_i \neq y_j$ and $z_k \neq y_j$ then $Z = OPT(i, j-1)$

**Proof.**

Symmetric to Case 2. $\square$

# Structure Corollary

## Corollary

obvious

part 1 of ST

$$
OPT(i,j) = \begin{cases} \varnothing & \text{if } i = 0 \text{ or } j = 0, \\ OPT(i-1,j-1) \circ x_i & \text{if } i,j > 0 \text{ and } x_i = y_j \\ \max(OPT(i,j-1), OPT(i-1,j)) & \text{if } i,j > 0 \text{ and } x_i \neq y_j \end{cases}
$$

parts 2,3 of ST

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Gives obvious recursive algorithm

- Can take exponential time (good exercise at home!)

Dynamic Programming!

- Top-Down: are problems getting "smaller"? What does "smaller" mean?
- Bottom-Up: two-dimensional table! What order to fill it in?

# Dynamic Programming Algorithm

```
LCS(X,Y) {
    for(i = 0 to m) M[i,0] = 0;
    for(j = 0 to n) M[0,j] = 0;
    for(i = 1 to m) {
        for(j = 1 to n) {
            if(x_i = y_j)
                M[i,j] = 1 + M[i-1,j-1];
            else
                M[i,j] = max(M[i,j-1], M[i-1,j]);
            }
        }
    return M[m,n];
}
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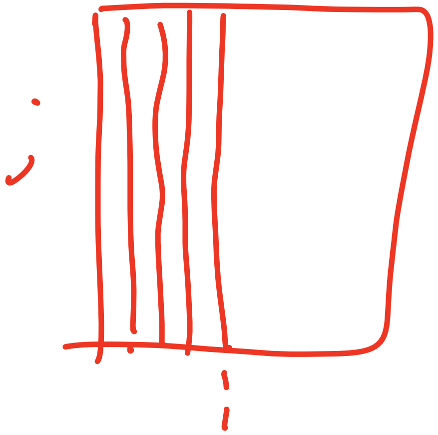

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            }
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    return $M[m, n]$;
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**Running Time:** $O(mn)$

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**Inductive Step:** Divide into three cases

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def alg    induction    ST

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3. If $x_i \neq y_j$, then

$$\begin{aligned}
M[i,j] &= \max(M[i,j-1], M[i-1,j]) && \text{(def of algorithm)} \\
&= \max(|OPT(i,j-1)|, |OPT(i-1,j)|) && \text{(induction)} \\
&= |OPT(i,j)| && \text{(structure thm/corollary)}
\end{aligned}$$

# Computing a Solution

Like we talked about last lecture: backtrack through dynamic programming table.

Details in CLRS 15.4

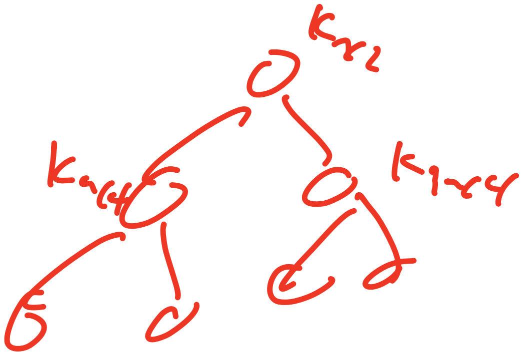
# Optimal Binary Search Trees

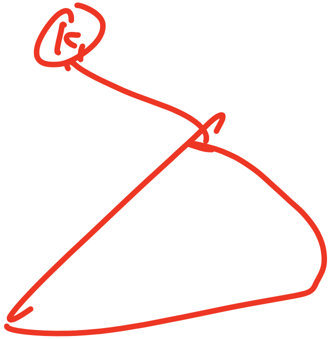
# Problem Definition

Input: probability distribution / search frequency of keys

- $n$ distinct keys $k_1 < k_2 < \cdots < k_n$
- For each $i \in [n]$, probability $p_i$ that we search for $k_i$ (so $\sum_{i=1}^{n} p_i = 1$)

What's the best binary search tree for these keys and frequencies?



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Cost of searching for $k_i$ in tree $T$ is $depth_T(k_i) + 1$ (say depth of root $= 0$)

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**Definition:** $c(T) = \sum_{i=1}^{n} p_i(depth_T(k_i) + 1)$

Problem: Find search tree $T$ minimizing cost.

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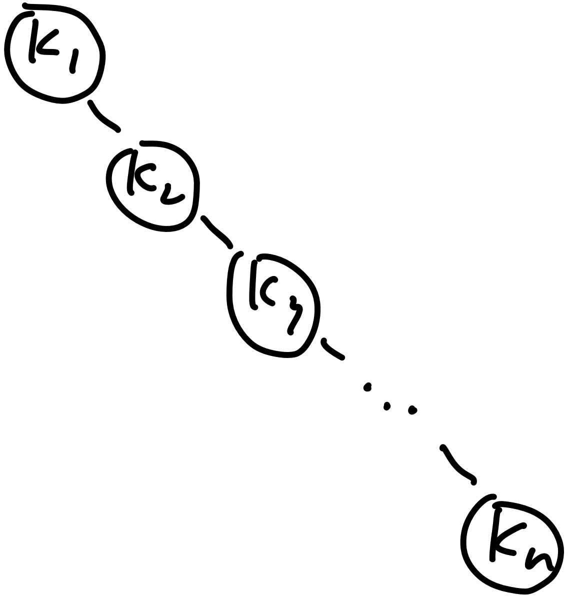

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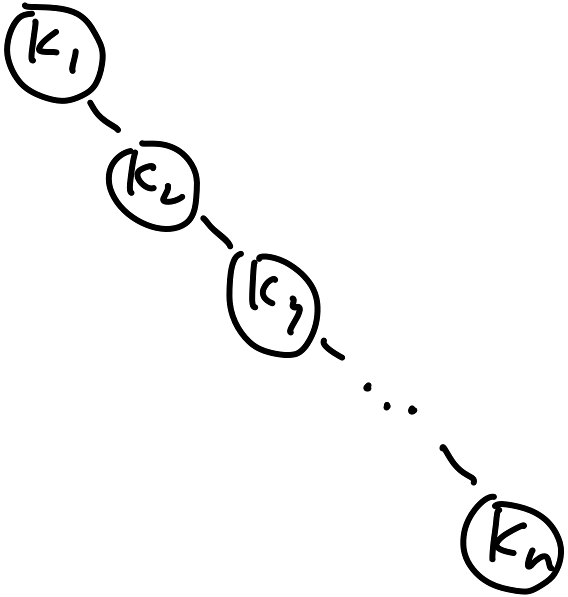


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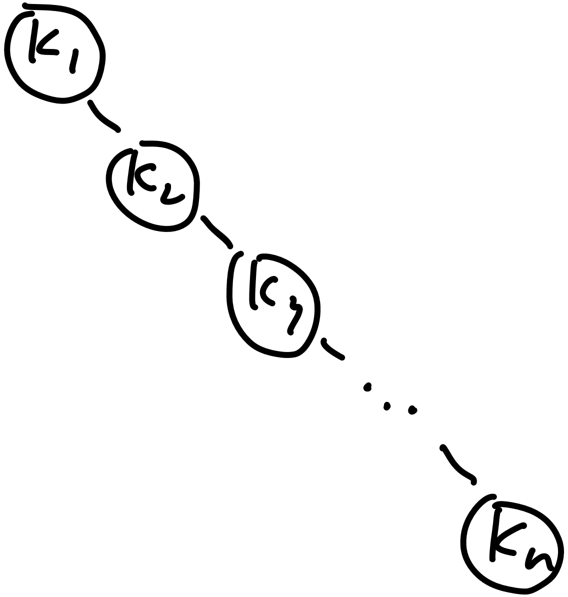


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Balanced search tree: $E[\text{cost}] \leq O(\log n)$

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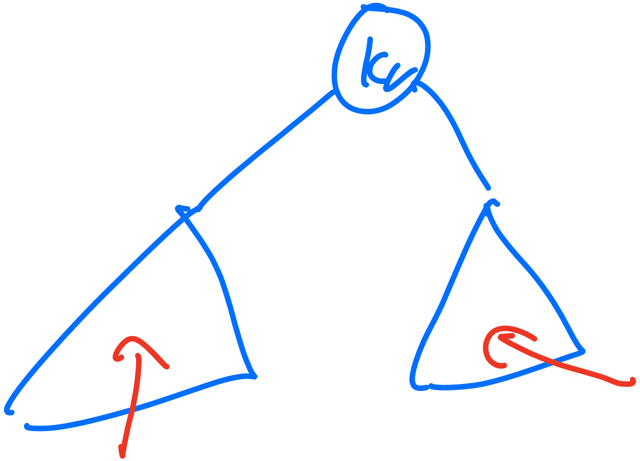
Suppose root is $\boldsymbol{k_r}$. What does optimal tree look like?




OPT for $k_{r+1}, \ldots, k_n$

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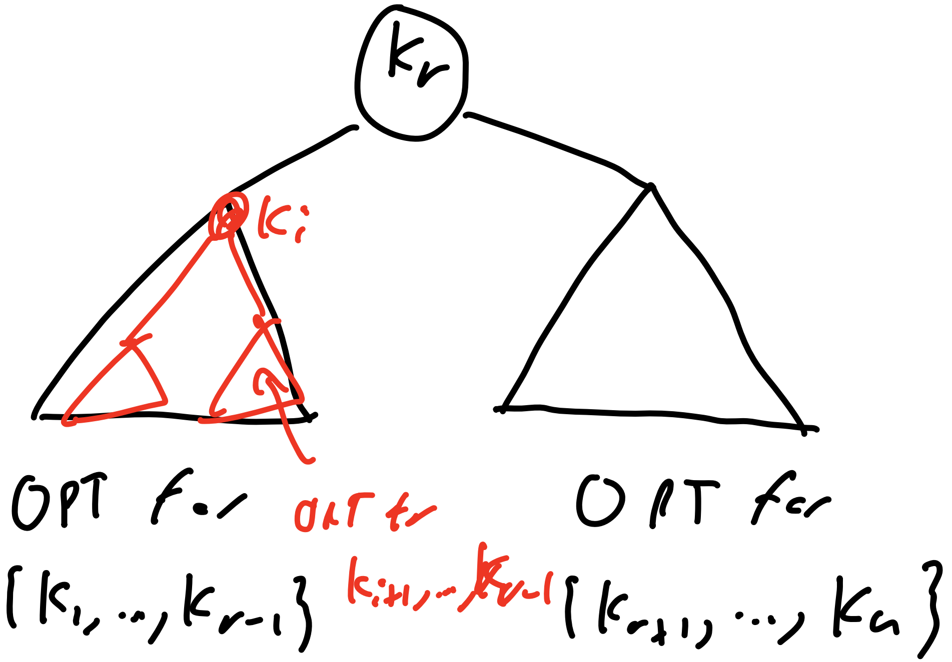

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Let $OPT(i,j)$ with $i \leq j$ be optimal tree for keys $\{k_i, k_{i+1}, \ldots, k_j\}$: tree $T$ minimizing $c(T) = \sum_{a=i}^{j} p_a(depth_T(k_a) + 1)$

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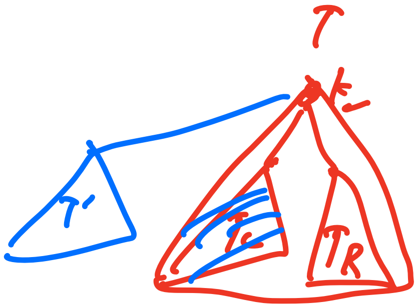
## Theorem (Optimal Substructure)

*Let $k_r$ be the root of $OPT(i,j)$. Then the left subtree of $OPT(i,j)$ is $OPT(i,r-1)$, and the right subtree of $OPT(i,j)$ is $OPT(r+1,j)$.*

# Proof Sketch of Optimal Substructure

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- Let $\boldsymbol{T} = \boldsymbol{OPT(i,j)}$, $\boldsymbol{T_L}$ its left subtree, $\boldsymbol{T_R}$ its right subtree.
- Suppose for contradiction $\boldsymbol{T_L \neq OPT(i, r-1)}$, let $\boldsymbol{T' = OPT(i, r-1)}$

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- Let $\boldsymbol{\hat{T}}$ be tree get by replacing $\boldsymbol{T_L}$ with $\boldsymbol{T'}$

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Whole bunch of math (see lecture notes): get that $c(\hat{T}) < c(T)$

Contradicts $T = OPT(i,j)$

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Symmetric argument works for $T_R = OPT(r+1, j)$

# Cost Corollary

## Corollary

$c(OPT(i,j)) = \sum_{a=i}^{j} p_a + \min_{i \le r \le j}(c(OPT(i,r-1)) + c(OPT(r+1,j)))$

Let $k_r$ be root of $OPT(i,j)$

$$
\begin{aligned}
c(OPT(i,j)) &= \sum_{a=i}^{j} p_a(depth_{OPT(i,j)}(k_a) + 1) \\
&= \sum_{a=i}^{r-1}(p_a(depth_{OPT(i,r-1)}(k_a) + 2)) + p_r + \sum_{a=r+1}^{j} p_a(depth_{OPT(r+1,j)}(k_a) + 2) \\
&= \sum_{a=i}^{j} p_a + \sum_{a=i}^{r-1}(p_a(depth_{OPT(i,r-1)}(k_a) + 1)) + \sum_{a=r+1}^{j} p_a(depth_{OPT(r+1,j)}(k_a) + 1) \\
&= \sum_{a=i}^{j} p_a + c(OPT(i,r-1)) + c(OPT(r+1,j)).
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&= \sum_{a=i}^{j} p_a + c(OPT(i,r-1)) + c(OPT(r+1,j)).
\end{aligned}
$$

Same logic holds for any possible root $\implies$ take min

# Algorithm

Fill in table $M$:

$$M[i,j] = \begin{cases} 0 & \text{if } i > j \\ \min_{i \le r \le j} \left( \sum_{a=i}^{j} p_a + M[i, r-1] + M[r+1, j] \right) & \text{if } i \le j \end{cases}$$

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Top-Down (memoization): are problems getting smaller?

# Algorithm

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**Correctness.** Claim $M[i,j] = c(OPT(i,j))$. Induction on $j - i$.

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- Base case: if $j - i < 0$ then $M[i,j] = OPT(i,j) = 0$

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- Base case: if $j - i < 0$ then $M[i,j] = OPT(i,j) = 0$
- Inductive step:

$$\begin{aligned} M[i,j] &= \min_{i \le r \le j} \left( \sum_{a=i}^{j} p_a + M[i,r-1] + M[r+1,j] \right) && \text{(alg def)} \\ &= \min_{i \le r \le j} \left( \sum_{a=i}^{j} p_a + c(OPT(i,r-1)) + c(OPT(r+1,j)) \right) && \text{(induction)} \\ &= c(OPT(i,j)) && \text{(cost corollary)} \end{aligned}$$

# Algorithm: Bottom-up

What order to fill the table in?

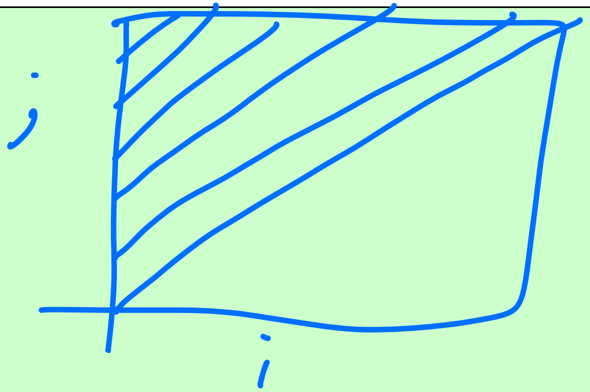
- Obvious approach: for($\boldsymbol{i} = \mathbf{1}$ to $\boldsymbol{n} - \mathbf{1}$) for($\boldsymbol{j} = \boldsymbol{i} + \mathbf{1}$ to $\boldsymbol{n}$) Doesn't work!

# Algorithm: Bottom-up

What order to fill the table in?

- Obvious approach: for($i = 1$ to $n - 1$) for($j = i + 1$ to $n$) Doesn't work!
- Take hint from induction: $j - i$

OBST {<br>
    Set $M[i,j] = 0$ for all $j > i$;<br>
    Set $M[i,i] = p_i$ for all $i$<br>
    for($\ell = 1$ to $n - 1$) {<br>
        for($i = 1$ to $n - \ell$) {<br>
            $j = i + \ell$<br>
            $M[i,j] = \min_{i \le r \le j}\left(\sum_{a=i}^{j} p_a + M[i, r-1] + M[r+1, j]\right)$;<br>
        }<br>
    }<br>
    return $M[1,n]$;<br>
}



# Analysis

**Correctness:** same as top-down

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- # table entries:

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# Analysis

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**Running Time:**

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- Time to compute table entry $M[i,j]$:

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**Running Time:**

- # table entries: $O(n^2)$
- Time to compute table entry $M[i,j]$: $O(j-i) = O(n)$

# Analysis

**Correctness:** same as top-down

**Running Time:**

- # table entries: $O(n^2)$
- Time to compute table entry $M[i,j]$: $O(j-i) = O(n)$

Total running time: $O(n^3)$

# Bonus Content

**Obvious Question:** Robustness.

- What if given distribution is *wrong*?

Want algorithm that gives a solution with cost a function of true optimal cost, "distance" between given distribution and true distribution.

Dinitz, Im, Lavastida, Moseley, Niaparast, Vassilvitskii. *Binary Search Trees with Distributional Predictions*. NeurIPS '24