Lecture 13: Basic Graph Algorithms

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October 8, 2024

601.433/633 Introduction to Algorithms

Introduction

Next 3-4 weeks: graphs!

- Super important abstractions, used all over the place in CS
- Most of my research is in graph algorithms (particularly when graph represents computer/communication network)
- Great course on Graph Theory in AMS

Today: review of basic graph algorithms from Data Structures, possibly one or two new

- Going to move pretty quickly, since much review: see CLRS for details!

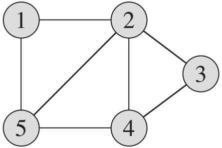
Basic Definitions

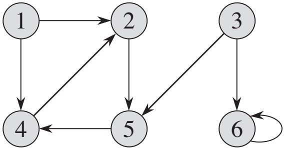
Definition

A graph $G = (V, E)$ is a pair where V is a set and $E \subseteq \binom{V}{2}$ (unordered pairs) or $E \subseteq V \times V$ (ordered pairs).

Notation:

- Elements of V are called *vertices* or *nodes*
- Elements of E are called *edges* or *arcs*.
- If $E \subseteq \binom{V}{2}$ then graph is *undirected*, if $E \subseteq V \times V$ graph is *directed*
- $|V| = n$ and $|E| = m$ (usually)
- So "size of input" $= n + m$





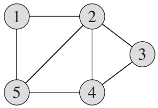
Representations

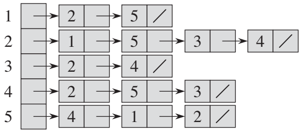
Adjacency List:

- Array A of length n
- $A[v]$ is linked list of vertices *adjacent* to v (edge from u to v)

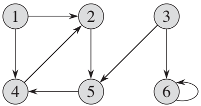
Adjacency Matrix:

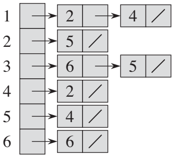
- $A \in \{0,1\}^{n \times n}$
- $A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$





	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0





	1	2	3	4	5	6
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Any way to improve these?

- Replace adjacency *list* with adjacency *structure*: Red-black tree, hash table, etc.
- Not traditional, doesn't gain us much, and more complicated. But better!

Breadth-First Search (BFS)

BFS Definition

Idea: explore graph in *levels* or *layers* from source s

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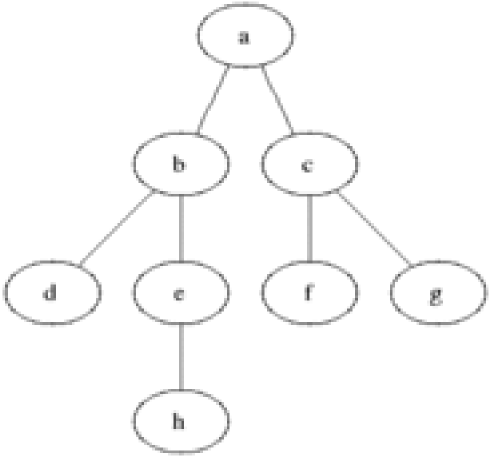

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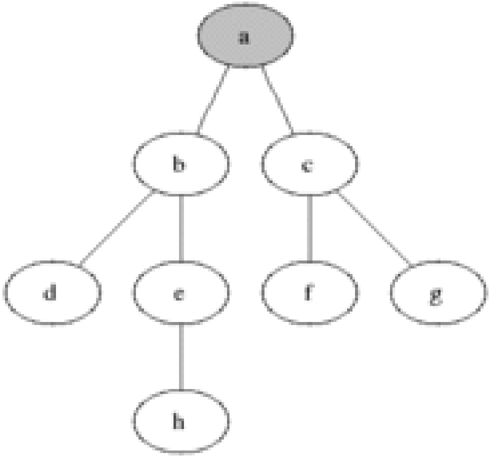

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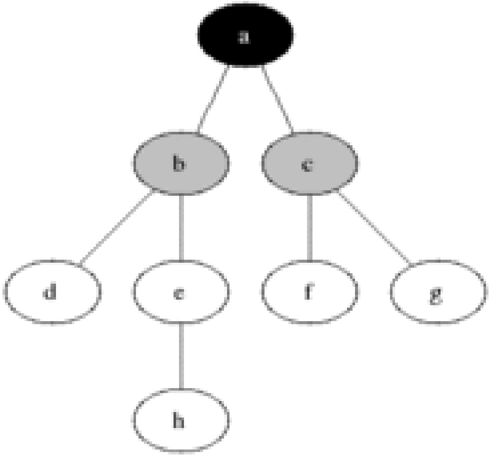

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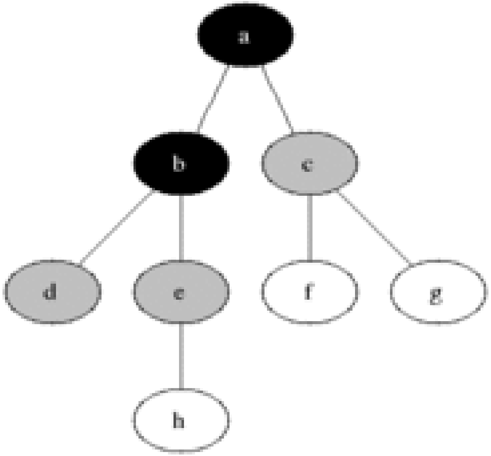

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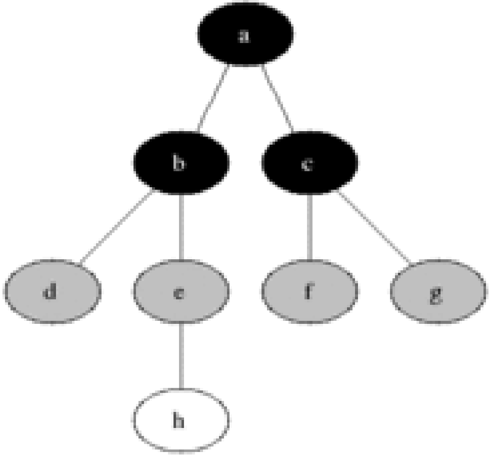

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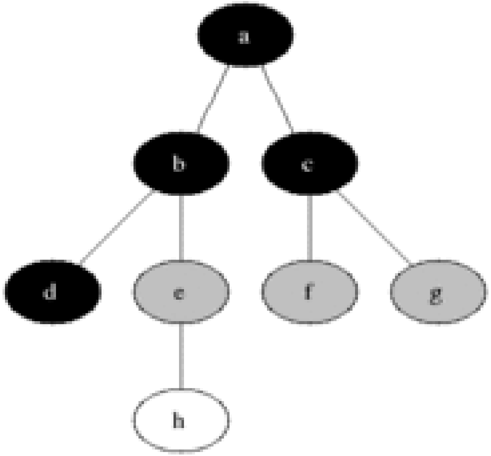

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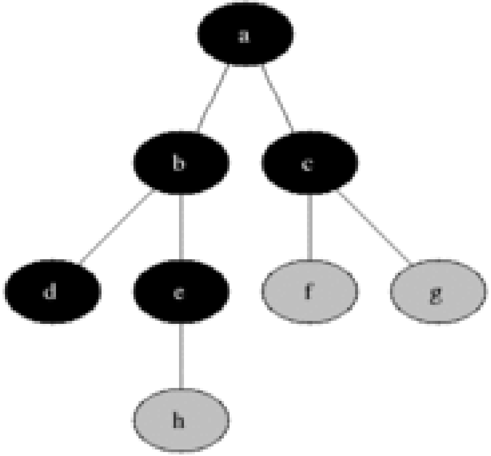

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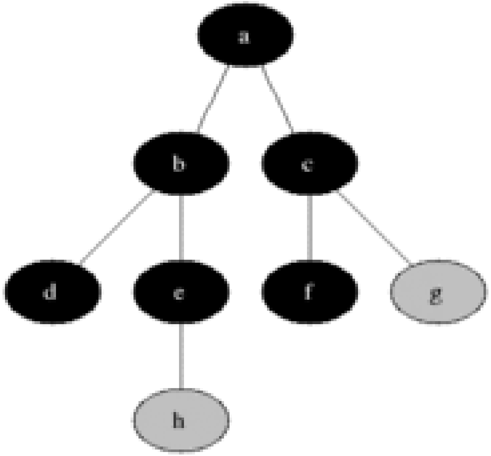

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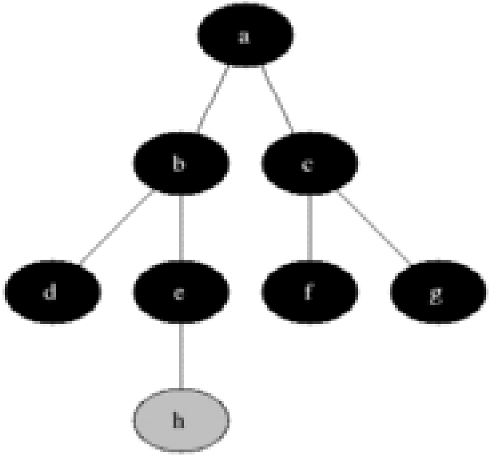

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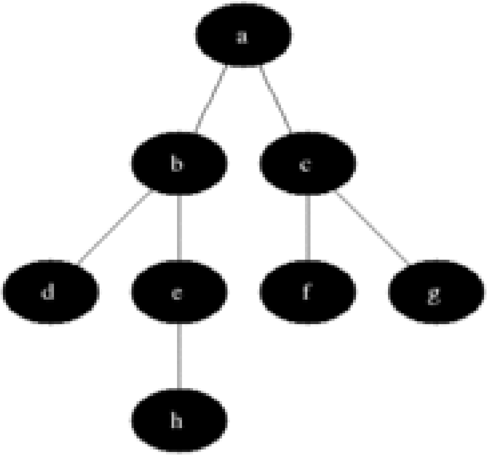

BFS Pseudocode

Idea: explore with a queue (FIFO)

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BFS(G = (V, E), s) {
    Set mark(s) = True;
    Set mark(v) = False for all v ∈ V ∖ {s};
    Enqueue(s);
    while(queue not empty) {
        v = Dequeue();
        forall neighbors u of v {
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  - Examine every edge twice: when each endpoint dequeued
  - Or (equivalent): Adjacency list scanned only when vertex dequeued

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Note: edges that cause a node to be enqueued form a tree!

Correctness / Shortest Paths

Definition: Distance $d(u, v)$ from u to v is min # edges in any path from u to v

Theorem (informal): BFS(s) gives shortest paths from s to all other nodes

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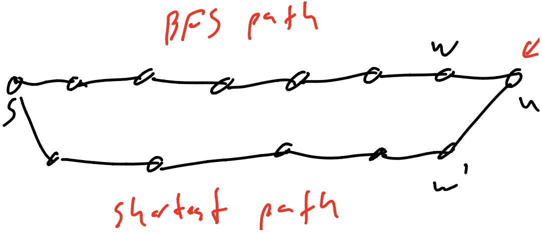
Proof Sketch:
Assume false for contradiction, let u be closest node to s where BFS(s) doesn't give shortest path



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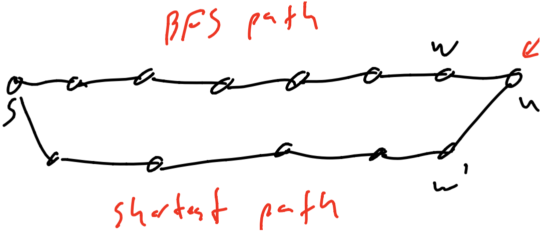

$d(s, w') < d(s, w)$

$\implies$ w' dequeued before w (since w' has correct distance by def of u)

$\implies$ u will be enqueued from w', not w. Contradiction.

Depth-First Search (DFS)

DFS: Definition

Intuition: Instead of exploring wide (breadth), explore far (deep): just keep walking until see a node we've already seen, then backtrack!

Init: for each $v \in V$, $mark(v) = False$;

DFS(v) {
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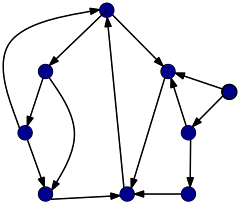

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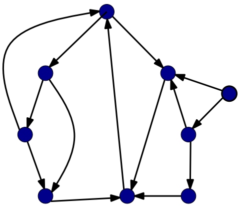

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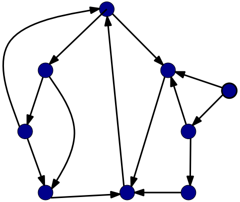

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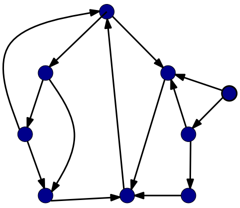

Running time: $O(m+n)$

- $O(n)$ initialization
- Every edge considered at most twice

DFS: Correctness

Definition: $\boldsymbol{u}$ is *reachable* from $\boldsymbol{v}$ if there is a path $\boldsymbol{v} = \boldsymbol{v_0}, \boldsymbol{v_1}, \ldots, \boldsymbol{v_k} = \boldsymbol{u}$ such that $(\boldsymbol{v_i}, \boldsymbol{v_{i+1}}) \in \boldsymbol{E}$ for all $\boldsymbol{i} \in \{0, 1, \ldots, \boldsymbol{k} - 1\}$.

Theorem

When DFS($\boldsymbol{v}$) terminates, it has visited (marked) all nodes that are reachable from $\boldsymbol{v}$.

Proof.

Suppose $\boldsymbol{u}$ reachable from $\boldsymbol{v}$ but not marked when DFS($\boldsymbol{v}$) terminates.

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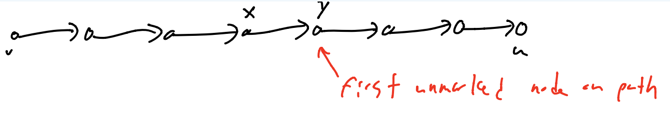

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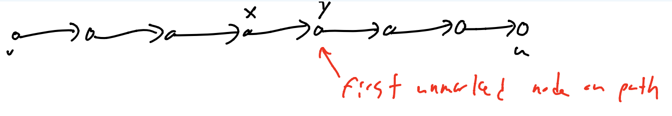

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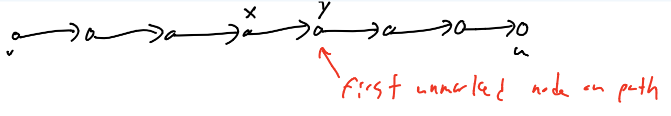

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$\implies$ y was either marked or DFS(y) called and it became marked.

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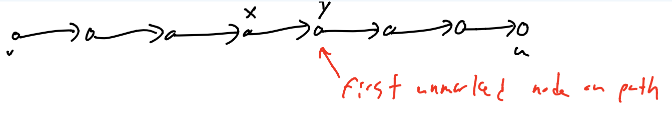

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Contradiction. □

Graph variant

After DFS($\boldsymbol{v}$), node marked if and only if reachable from $\boldsymbol{v}$.

Might want to continue until all nodes marked.

```
DFS(G) {
    for all v ∈ V, set mark(v) = False;
    while there exists an unmarked node v {
        DFS(v);
    }
}
```

Timestamps

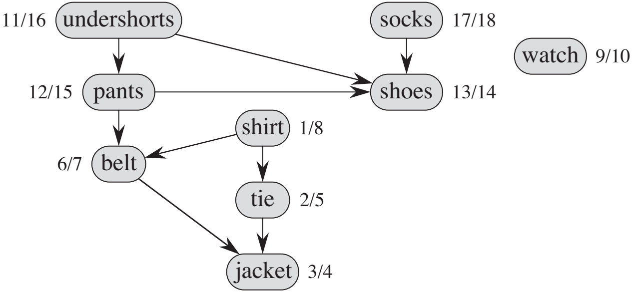
Explicitly keep track of "start" and "finishing" times

- Replaces ***mark***

```
DFS(G) {
    t = 0;
    for all v ∈ V {
        start(v) = 0;
        finish(v) = 0;
    }
    while ∃v ∈ V with start(v) = 0 {
        DFS(v);
    }
}
```

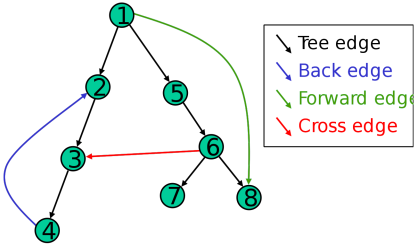
```
DFS(v) {
    t = t + 1;
    start(v) = t;
    for each edge (v, u) ∈ A[v] {
        if start(u) == 0 then DFS(u);
    }
    t = t + 1;
    finish(v) = t;
}
```

Timestamp Example



Edge Types

DFS naturally gives a spanning forest: edge (v, u) if DFS(v) calls DFS(u)



Forward Edges: (v, u) such that u descendent of v (includes tree edges)

Back Edges: (v, u) such that u an ancestor of v

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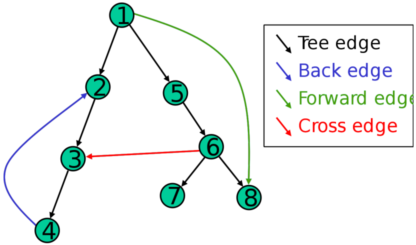

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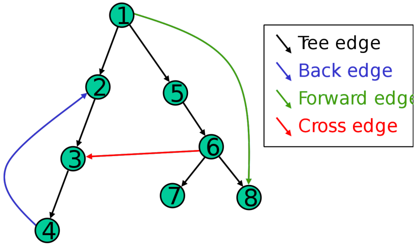

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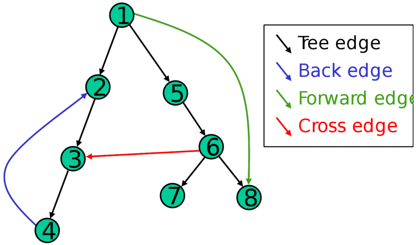

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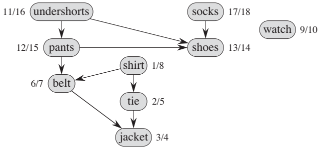
Topological Sort

Definitions

Definition

A directed graph G is a *Directed Acyclic Graph (DAG)* if it has no directed cycles.



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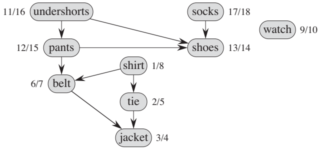

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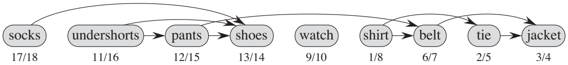
A *topological sort* $v_1, v_2, \ldots, v_n$ of a DAG is an ordering of the vertices such that all edges are of the form (v_i, v_j) with $i < j$.



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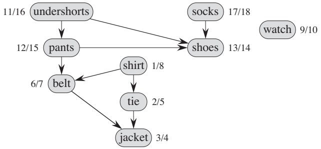

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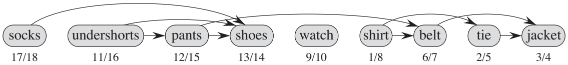

Q: Can we always topological sort a DAG? How fast?

Topological Sort

Algorithm (informal): Run DFS($\boldsymbol{G}$). When DFS($\boldsymbol{v}$) returns, put $\boldsymbol{v}$ at beginning of list

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```
DFS(G) {
    list → head = NULL;
    t = 0;
    for all v ∈ V {
        start(v) = 0;
        finish(v) = 0;
    }
    while ∃v ∈ V with start(v) = 0 {
        DFS(v);
    }
}
```

```
DFS(v) {
    t = t + 1;
    start(v) = t;
    for each edge (v, u) ∈ A[v] {
        if start(u) == 0 then DFS(u);
    }
    t = t + 1;
    finish(v) = t;
    temp = list → head;
    list → head = v;
    list → head → next = temp;
}
```

Characterizing DAGs

Theorem

A directed graph $\boldsymbol{G}$ is a DAG if and only if DFS($\boldsymbol{G}$) has no back edges.

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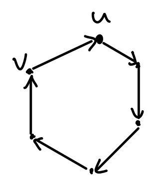
If ($\Leftarrow$): contrapositive. If G has a directed cycle C:

- Let $u \in C$ with minimum start value, v predecessor in cycle
- All nodes in C reachable from $u \implies$ all nodes in C descendants of u
- (v, u) a back edge



$\square$

Topological Sort Analysis

Correctness: Since G a DAG, never see back edge

$\implies$ Every edge (v, u) out of v a forward or cross edge

$\implies$ $finish(u) < finish(v)$

$\implies$ u already in list when v added to beginning

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Running Time: Same as DFS! $O(m + n)$