Lecture 15: Single-Source Shortest Paths

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601.433/633 Introduction to Algorithms

Introduction

Setup:

- Directed graph $G = (V, E)$
- Length $\ell(x, y)$ on each edge $(x, y) \in E$ (equivalent: $\ell : E \to \mathbb{R}$)
- Length of path P is $\ell(P) = \sum_{(x,y)\in P} \ell(x, y)$
- $d(x, y) = \min_{x \to y \text{ paths } P} \ell(P)$

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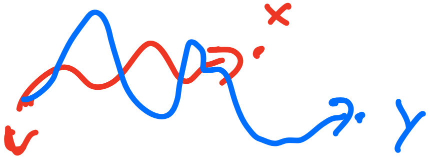
Today: source $v \in V$, want to compute shortest path from v to every $u \in V$

- $d(u) = d(v,u)$ for all $u \in V$
- Representation: "shortest path tree" out of v.
- Often only care about distances – can reconstruct tree from distances.

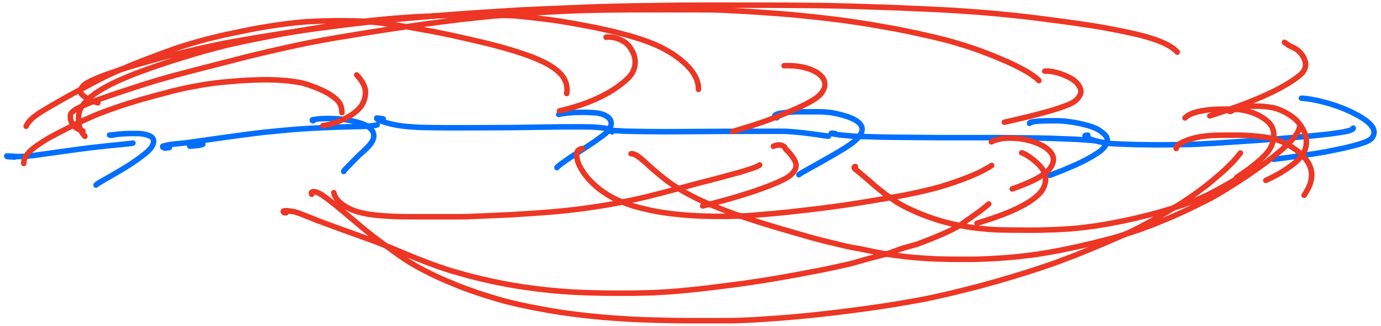


Bellman-Ford

Dynamic Programming Approach

Subproblems:

- $OPT(u, i)$: shortest path from v to u that uses at most i hops (edges)
- If no such path, set to "infinitely long" fake path.
- For simplicity, create loop (edge to and from the same node) at every node, length $\mathbf{0}$



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Theorem (Optimal Substructure)

$$\ell(OPT(u,k)) = \begin{cases} 0 & \textit{if } u = v, k = 0 \\ \infty & \textit{if } u \neq v, k = 0 \\ & \textit{otherwise} \end{cases}$$

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Theorem (Optimal Substructure)

$$\ell(OPT(u,k)) = \begin{cases} 0 & \text{if } u = v, k = 0 \\ \infty & \text{if } u \neq v, k = 0 \\ \min_{w:(w,u)\in E} \left(\ell(OPT(w,k-1)) + \ell(w,u)\right) & \text{otherwise} \end{cases}$$

$\min_{w} \left(\ell(OPT(w)) + \ell(w,u) \right)$

Proof of Optimal Substructure

Induction on $\boldsymbol{k}$.

$\boldsymbol{k = 0 :}$ ✓. So let $\boldsymbol{k \geq 1}$.

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- $\leq$: Let $x = \arg\min_{w:(w,u)\in E}(\ell(OPT(w, k-1)) + \ell(w, u))$
  $\implies OPT(x, k-1) \circ (x, u)$ is a $v \to u$ path with at most k edges, length $\ell(OPT(x, k-1)) + \ell(x, u))$
  $\implies \ell(OPT(u, k)) \leq \min_{w:(w,u)\in E}(\ell(OPT(w, k-1)) + \ell(w, u))$

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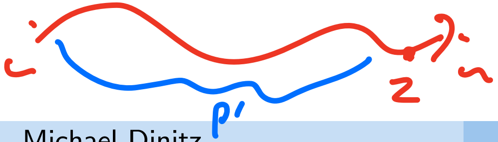
  $\implies \ell(OPT(u,k)) \leq \min_{w:(w,u)\in E} (\ell(OPT(w,k-1)) + \ell(w,u))$
- $\geq$: Let z be node before u in $OPT(u,k)$, and let P' be the first $k-1$ edges of $OPT(u,k)$. Then

$$\begin{aligned}\ell(OPT(u,k)) &= \ell(P') + \ell(z,u) \geq \ell(OPT(z,k-1)) + \ell(z,u) \\ &\geq \min_{w:(w,u)\in E} (\ell(OPT(w,k-1)) + \ell(w,u))\end{aligned}$$



Bellman-Ford Algorithm

Obvious dynamic program!

```
$M[u,0] = \infty$ for all $u \in V, u \neq v$
$M[v,0] = 0$

for($k = 1$ to $n-1$) {
    for($u \in V$) {
        $M[u,k] = \min_{w:(w,u)\in E}(M[w,k-1] + \ell(w,u))$
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- Obvious: $O(n^3)$

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for($k = 1$ to $n - 1$) {          $O(n)$
    for($u \in V$) {
        $M[u,k] = \min_{w:(w,u) \in E}(M[w,k-1] + \ell(w,u))$          $O(m)$
    }
}
```

Running Time:

- Obvious: $O(n^3)$
- Smarter: $O(mn)$

Bellman-Ford: Correctness

Theorem

After algorithm completes, $\boldsymbol{M[u,k] = \ell(OPT(u,k))}$ *for all* $\boldsymbol{k \le n-1}$ *and* $\boldsymbol{u \in V}$.

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After algorithm completes, $M[u,k] = \ell(OPT(u,k))$ *for all* $k \le n-1$ *and* $u \in V$*.*

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$$\begin{aligned}
M[u,k] &= \min_{w:(w,u)\in E} (M[w,k-1]) + \ell(w,u)) && \text{(algorithm)} \\
&= min_{w:(w,u)\in E}(\ell(OPT(w,k-1)) + \ell(w,u)) && \text{(induction)} \\
&= \ell(OPT(u,k)) && \text{(optimal substructure)}
\end{aligned}$$

□

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Fun fact: best-known algorithm with negative (real) edge weights until this year!

Jeremy Fineman. *Single-Source Shortest Paths with Negative Real Weights in* $\tilde{O}(mn^{8/9})$ *Time.*
STOC '24

Relaxations

Common primitive in shortest path algorithms

- Reinterpret Bellman-Ford via relaxations
- Use relaxations for Dijkstra's algorithm

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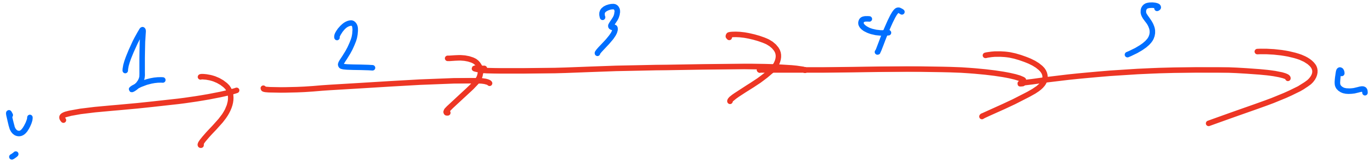
Intuition for relax(x, y): can we improve $\hat{d}(y)$ by going through x?

```
relax($x, y$) {
    if($\hat{d}(y) > \hat{d}(x) + \ell(x, y)$) {
        $\hat{d}(y) = \hat{d}(x) + \ell(x, y)$
        y.parent = x
    }
}
```

Bellman-Ford as Relaxations

```
for($i = 1$ to $n$) {
    foreach($u \in V$) {
        foreach(edge $(x, u)$) {
            relax($x, u$)
        }
    }
}
```



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Not precisely the same: freezing/parallelism

Dijkstra's Algorithm

High Level

Intuition: “greedy starting at v”

- BFS but with edge lengths: use priority queue (heap) instead of queue!

Pros: faster than Bellman-Ford (super fast with appropriate data structures)

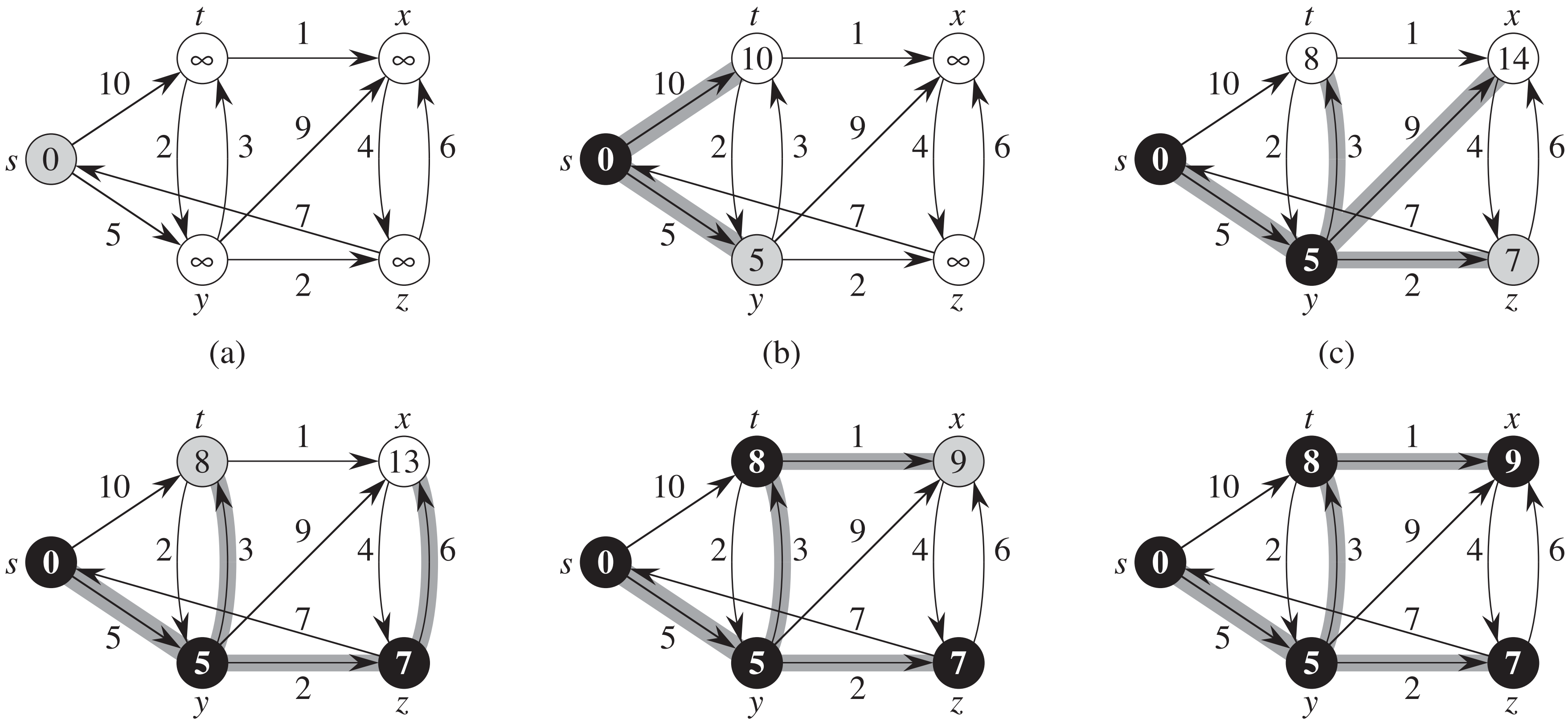
Cons: Doesn’t work with negative edge weights.

Dijkstra's Algorithm

```
$T = \varnothing$
$\hat{d}(v) = 0$
$\hat{d}(u) = \infty$ for all $u \neq v$

while(not all nodes in $T$) {
    let $u$ be node not in $T$ with minimum $\hat{d}(u)$
    Add $u$ to $T$
    foreach edge $(u, x)$ with $x \notin T$ {
        relax(u,x)
    }
}
```

Dijkstra Example



(a) (b) (c)

Dijkstra Correctness

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Throughout the algorithm:

1. T *is a shortest-path tree from* v *to the nodes in* T*, and*
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Proof. Induction on $|T|$ (iterations of algorithm)

Base Case: After first iteration (when $|T| = 1$), added v to T with $\hat{d}(v) = d(v) = 0$ ✓

Correctness: Inductive Step (Sketch)

Consider iteration when $\boldsymbol{u}$ added to $\boldsymbol{T}$, let $\boldsymbol{w} = \boldsymbol{u.parent}$

$$\implies \hat{d}(u) \underset{\substack{\uparrow \\ \text{parent def}}}{=} \hat{d}(w) + \ell(w,u) \underset{\substack{\uparrow \\ \text{induction}}}{=} d(w) + \ell(w,u) \text{ (induction)}$$

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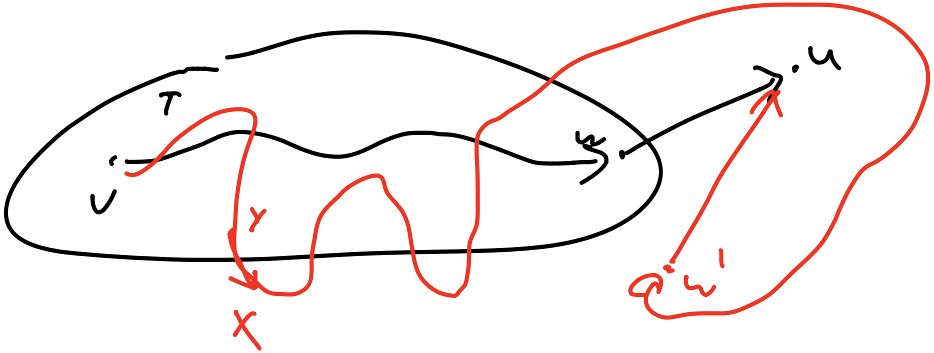

- Red path P actual shortest path, black path found by Dijkstra
- w' predecessor of u on P. Can't be in T.
  - If it was, would have $\hat{d}(w') = d(w')$ by induction, would have relaxed (w', u), so would have $w' = u.parent$
- x first node of P outside T, previous node y

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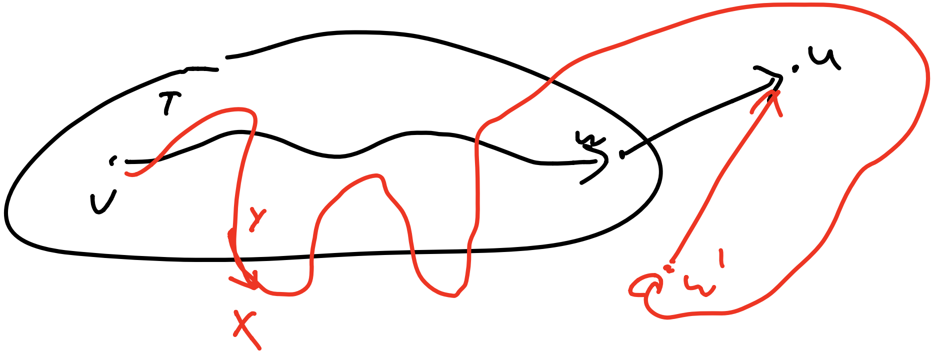

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- $\boldsymbol{x}$ first node of $\boldsymbol{P}$ outside $\boldsymbol{T}$, previous node $\boldsymbol{y}$

$$\hat{\boldsymbol{d}}(\boldsymbol{x}) \leq \hat{\boldsymbol{d}}(\boldsymbol{y}) + \ell(\boldsymbol{y}, \boldsymbol{x}) = \boldsymbol{d}(\boldsymbol{y}) + \ell(\boldsymbol{y}, \boldsymbol{x}) < \ell(\boldsymbol{P}) = \boldsymbol{d}(\boldsymbol{u}) \leq \hat{\boldsymbol{d}}(\boldsymbol{u})$$

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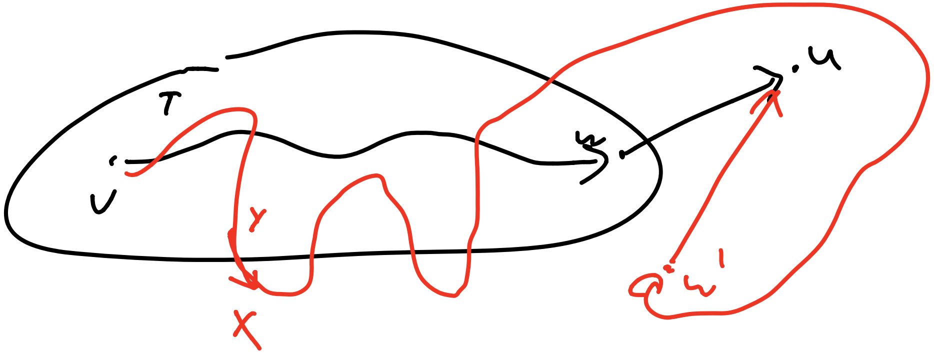

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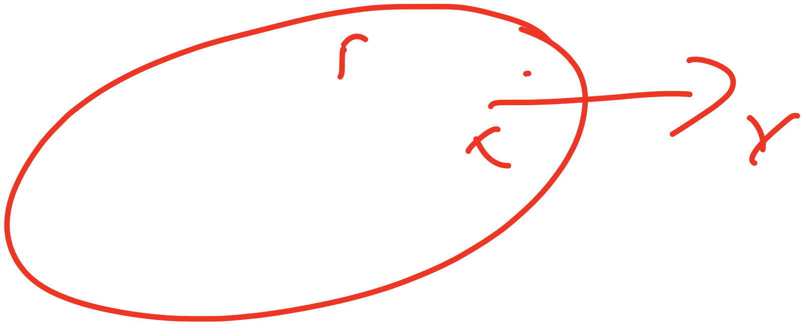
$$\hat{d}(x) \le \hat{d}(y) + \ell(y,x) = d(y) + \ell(y,x) < \ell(P) = d(u) \le \hat{d}(u)$$

Contradiction! Algorithm would have chosen $\boldsymbol{x}$ next, not $\boldsymbol{u}$.

Running Time

Algorithm needs to:

- Select node with minimum $\hat{\boldsymbol{d}}$ value $\boldsymbol{n}$ times
- Decrease a $\hat{\boldsymbol{d}}$ value at most once per relaxation $\implies \le \boldsymbol{m}$ times.



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Keep $\hat{d}$ values in a heap!

- Insert n times
- Extract-Min n times
- Decrease-Key m times

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Fibonacci Heap:

- Insert, Decrease-Key $O(1)$ amortized
- Extract-Min $O(\log n)$ amortized

$\implies O(m + n\log n)$ running time