Lecture 15: Single-Source Shortest Paths

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601.433/633 Introduction to Algorithms

Introduction

Setup:

- Directed graph $G = (V, E)$
- Length $\ell(x, y)$ on each edge $(x, y) \in E$ (equivalent: $\ell : E \to \mathbb{R}$)
- Length of path P is $\ell(P) = \sum_{(x,y) \in P} \ell(x, y)$
- $d(x, y) = \min_{x \to y \text{ paths } P} \ell(P)$

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Today: source $v \in V$, want to compute shortest path from v to every $u \in V$

- $d(u) = d(v, u)$ for all $u \in V$
- Representation: "shortest path tree" out of v.
- Often only care about distances – can reconstruct tree from distances.

Bellman-Ford

Dynamic Programming Approach

Subproblems:

- $OPT(u, i)$: shortest path from v to u that uses at most i hops (edges)
- If no such path, set to "infinitely long" fake path.
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Theorem (Optimal Substructure)

$$\ell(OPT(u,k)) = \begin{cases} 0 & \text{if } u = v, k = 0 \\ \infty & \text{if } u \neq v, k = 0 \\ & \text{otherwise} \end{cases}$$

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$$\ell(OPT(u,k)) = \begin{cases} 0 & \textit{if } u = v, k = 0 \\ \infty & \textit{if } u \neq v, k = 0 \\ \min_{w:(w,u)\in E} (\ell(OPT(w, k-1)) + \ell(w,u)) & \textit{otherwise} \end{cases}$$

Proof of Optimal Substructure

Induction on $\boldsymbol{k}$.

$\boldsymbol{k} = \mathbf{0}:$ ✓. So let $\boldsymbol{k} \geq \mathbf{1}$.

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$\leq$: Let $x = \arg\min_{w:(w,u)\in E}(\ell(OPT(w, k-1)) + \ell(w, u))$

$\implies$ $OPT(x, k-1) \circ (x, u)$ is a $v \to u$ path with at most k edges, length $\ell(OPT(x, k-1)) + \ell(x, u))$

$\implies$ $\ell(OPT(u, k)) \leq \min_{w:(w,u)\in E}(\ell(OPT(w, k-1)) + \ell(w, u))$

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  $\implies \ell(OPT(u, k)) \leq \min_{w:(w,u)\in E}(\ell(OPT(w, k-1)) + \ell(w, u))$
- $\geq$: Let z be node before u in $OPT(u, k)$, and let P' be the first $k-1$ edges of $OPT(u, k)$. Then

$$\begin{aligned}\ell(OPT(u,k)) &= \ell(P') + \ell(z,u) \geq \ell(OPT(z,k-1)) + \ell(z,u) \\ &\geq \min_{w:(w,u)\in E}(\ell(OPT(w,k-1)) + \ell(w,u))\end{aligned}$$

Bellman-Ford Algorithm

Obvious dynamic program!

$M[u,0] = \infty$ for all $u \in V, u \neq v$
$M[v,0] = 0$

for($k = 1$ to $n-1$) {
    for($u \in V$) {
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Running Time:

- Obvious: $O(n^3)$
- Smarter: $O(mn)$

Bellman-Ford: Correctness

Theorem

After algorithm completes, $M[u, k] = \ell(OPT(u, k))$ *for all* $k \leq n - 1$ *and* $u \in V$.

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$$\begin{aligned} M[u,k] &= \min_{w:(w,u)\in E} (M[w,k-1]) + \ell(w,u)) && \text{(algorithm)} \\ &= min_{w:(w,u)\in E} (\ell(OPT(w,k-1)) + \ell(w,u)) && \text{(induction)} \\ &= \ell(OPT(u,k)) && \text{(optimal substructure)} \end{aligned}$$

□

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Fun fact: best-known algorithm with negative (real) edge weights until this year!

Jeremy Fineman. *Single-Source Shortest Paths with Negative Real Weights in* $\tilde{O}(mn^{8/9})$ *Time*. STOC '24

Relaxations

Common primitive in shortest path algorithms

- Reinterpret Bellman-Ford via relaxations
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Intuition for relax(x, y): can we improve $\hat{d}(y)$ by going through x?

```
relax(x, y) {
    if(d̂(y) > d̂(x) + ℓ(x, y)) {
        d̂(y) = d̂(x) + ℓ(x, y)
        y.parent = x
    }
}
```

Bellman-Ford as Relaxations

```
for(i = 1 to n) {
    foreach(u ∈ V) {
        foreach(edge (x, u)) {
            relax(x, u)
        }
    }
}
```

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Not precisely the same: freezing/parallelism

Dijkstra's Algorithm

High Level

Intuition: “greedy starting at v”

- BFS but with edge lengths: use priority queue (heap) instead of queue!

Pros: faster than Bellman-Ford (super fast with appropriate data structures)

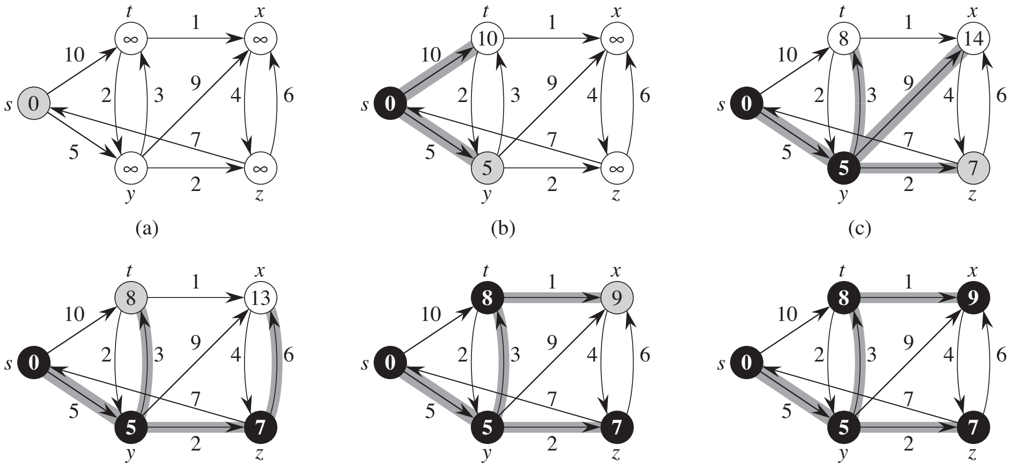
Cons: Doesn't work with negative edge weights.

Dijkstra's Algorithm

$\boldsymbol{T} = \varnothing$
$\hat{\boldsymbol{d}}(\boldsymbol{v}) = \mathbf{0}$
$\hat{\boldsymbol{d}}(\boldsymbol{u}) = \infty$ for all $\boldsymbol{u} \neq \boldsymbol{v}$

```
while(not all nodes in T) {
    let u be node not in T with minimum d̂(u)
    Add u to T
    foreach edge (u,x) with x ∉ T {
        relax(u,x)
    }
}
```

Dijkstra Example



Dijkstra Correctness

Theorem

Throughout the algorithm:

1. $\boldsymbol{T}$ *is a shortest-path tree from* $\boldsymbol{v}$ *to the nodes in* $\boldsymbol{T}$, *and*
2. $\hat{\boldsymbol{d}}(\boldsymbol{u}) = \boldsymbol{d}(\boldsymbol{u})$ *for every* $\boldsymbol{u} \in \boldsymbol{T}$.

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Proof. Induction on $|\boldsymbol{T}|$ (iterations of algorithm)

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Proof. Induction on $|T|$ (iterations of algorithm)

Base Case: After first iteration (when $|T| = 1$), added v to T with $\hat{d}(v) = d(v) = 0$ ✓

Correctness: Inductive Step (Sketch)

Consider iteration when $\boldsymbol{u}$ added to $\boldsymbol{T}$, let $\boldsymbol{w} = \boldsymbol{u.parent}$

$\implies \hat{\boldsymbol{d}}(\boldsymbol{u}) = \hat{\boldsymbol{d}}(\boldsymbol{w}) + \ell(\boldsymbol{w}, \boldsymbol{u}) = \boldsymbol{d}(\boldsymbol{w}) + \ell(\boldsymbol{w}, \boldsymbol{u})$ (induction)

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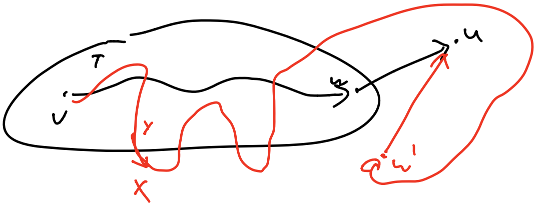

- Red path $\boldsymbol{P}$ actual shortest path, black path found by Dijkstra
- $\boldsymbol{w}'$ predecessor of $\boldsymbol{u}$ on $\boldsymbol{P}$. Can't be in $\boldsymbol{T}$.
  - If it was, would have $\hat{\boldsymbol{d}}(\boldsymbol{w}') = \boldsymbol{d}(\boldsymbol{w}')$ by induction, would have relaxed $(\boldsymbol{w}', \boldsymbol{u})$, so would have $\boldsymbol{w}' = \boldsymbol{u.parent}$
- $\boldsymbol{x}$ first node of $\boldsymbol{P}$ outside $\boldsymbol{T}$, previous node $\boldsymbol{y}$

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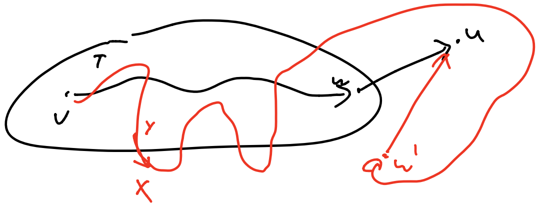

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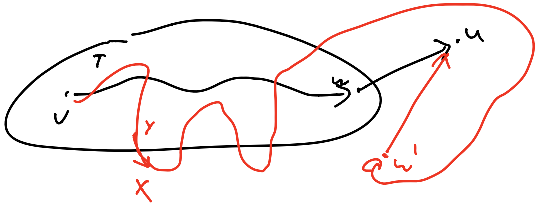

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Contradiction! Algorithm would have chosen x next, not u.

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Fibonacci Heap:

- Insert, Decrease-Key $O(1)$ amortized
- Extract-Min $O(\log n)$ amortized

$\implies O(m + n\log n)$ running time