Lecture 16: All-Pairs Shortest Paths

Michael Dinitz

October 24, 2024

601.433/633 Introduction to Algorithms

Announcements

- Mid-Semester feedback on Courselore!
- No lecture notes

Introduction

Setup:

- Directed graph $G = (V, E)$
- Length $\ell(x, y)$ on each edge $(x, y) \in E$
- Length of path P is $\ell(P) = \sum_{(x,y)\in P} \ell(x, y)$
- $d(x, y) = \min_{x \to y \text{ paths } P} \ell(P)$

Last time: All distances from source node $v \in V$.

Today: Distances between all pairs of nodes!

Introduction

Setup:

- Directed graph $G = (V, E)$
- Length $\ell(x, y)$ on each edge $(x, y) \in E$
- Length of path P is $\ell(P) = \sum_{(x,y)\in P} \ell(x, y)$
- $d(x, y) = \min_{x \to y \text{ paths } P} \ell(P)$

Last time: All distances from source node $v \in V$.

Today: Distances between all pairs of nodes!

Obvious solution:

Introduction

Setup:

- Directed graph $G = (V, E)$
- Length $\ell(x, y)$ on each edge $(x, y) \in E$
- Length of path P is $\ell(P) = \sum_{(x,y)\in P} \ell(x, y)$
- $d(x, y) = \min_{x \to y \text{ paths } P} \ell(P)$

Last time: All distances from source node $v \in V$.

Today: Distances between all pairs of nodes!

Obvious solution: single-source from each $v \in V$

Introduction

Setup:

- Directed graph $G = (V, E)$
- Length $\ell(x, y)$ on each edge $(x, y) \in E$
- Length of path P is $\ell(P) = \sum_{(x,y)\in P} \ell(x, y)$
- $d(x, y) = \min_{x \to y \text{ paths } P} \ell(P)$

Last time: All distances from source node $v \in V$.

Today: Distances between all pairs of nodes!

Obvious solution: single-source from each $v \in V$

- No negative weights: n runs of Dijkstra, time $O(n(m + n \log n))$
- Negative weights: n runs of Bellman-Ford, time $O(nmn) = O(mn^2)$

Introduction

Setup:

- Directed graph $G = (V, E)$
- Length $\ell(x, y)$ on each edge $(x, y) \in E$
- Length of path P is $\ell(P) = \sum_{(x,y)\in P} \ell(x, y)$
- $d(x, y) = \min_{x \to y \text{ paths } P} \ell(P)$

Last time: All distances from source node $v \in V$.

Today: Distances between all pairs of nodes!

Obvious solution: single-source from each $v \in V$

- No negative weights: n runs of Dijkstra, time $O(n(m + n \log n))$
- Negative weights: n runs of Bellman-Ford, time $O(nmn) = O(mn^2)$

Can we do better? Particularly for negative edge weights?

- Main goal today: Negative weights as fast as possible.

Floyd-Warshall Algorithm

Floyd-Warshall: A Different Dynamic Programming Approach

To simplify notation, let $V = \{1, 2, \ldots, n\}$ and $\ell(i, j) = \infty$ if $(i, j) \notin E$

Bellman-Ford subproblems: length of shortest path with at most some number of edges

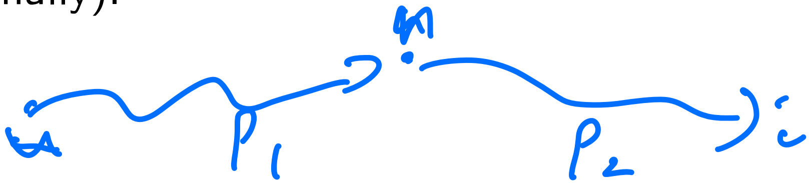
Floyd-Warshall: A Different Dynamic Programming Approach

To simplify notation, let $V = \{1, 2, \ldots, n\}$ and $\ell(i,j) = \infty$ if $(i,j) \notin E$

Bellman-Ford subproblems: length of shortest path with at most some number of edges

New subproblems:

- Intuition: "shortest path from u to v either goes through node n, or it doesn't"
  - If it doesn't: shortest uses only first nodes in $\{1, 2, \ldots, n-1\}$.
  - If it does: consists of a path P_1 from u to n and a path P_2 from n to v, neither of which uses n (internally).



Floyd-Warshall: A Different Dynamic Programming Approach

To simplify notation, let $V = \{1, 2, \ldots, n\}$ and $\ell(i, j) = \infty$ if $(i, j) \notin E$

Bellman-Ford subproblems: length of shortest path with at most some number of edges

New subproblems:

- Intuition: "shortest path from u to v either goes through node n, or it doesn't"
  - If it doesn't: shortest uses only first nodes in $\{1, 2, \ldots, n-1\}$.
  - If it does: consists of a path P_1 from u to n and a path P_2 from n to v, neither of which uses n (internally).
- Subproblems: shortest path from u to v that only uses nodes in $\{1, 2, \ldots k\}$ for all u, v, k.

Formalizing Subproblems

$u \to v$ path P: "intermediate nodes" are all nodes in P other than u, v.

d_{ij}^k: distance from i to j using only $i \to j$ paths with intermediate vertices in $\{1, 2, \ldots, k\}$.

- Goal: compute d_{ij}^k for all $i, j, k \in [n]$.
- Return d_{ij}^n for all $i, j \in V$.

Formalizing Subproblems

$u \to v$ path P: "intermediate nodes" are all nodes in P other than u, v.

d_{ij}^k: distance from i to j using only $i \to j$ paths with intermediate vertices in $\{1, 2, \ldots, k\}$.

- Goal: compute d_{ij}^k for all $i, j, k \in [n]$.
- Return d_{ij}^n for all $i, j \in V$.

$$d_{ij}^k = \begin{cases} & \text{if } k = 0 \\ & \text{if } k \geq 1 \end{cases}$$

Formalizing Subproblems

$u \to v$ path P: "intermediate nodes" are all nodes in P other than u, v.

d_{ij}^k: distance from i to j using only $i \to j$ paths with intermediate vertices in $\{1, 2, \ldots, k\}$.

- Goal: compute d_{ij}^k for all $i, j, k \in [n]$.
- Return d_{ij}^n for all $i, j \in V$.

$$d_{ij}^k = \begin{cases} \ell(i,j) & \text{if } k = 0 \\ & \text{if } k \geq 1 \end{cases}$$

d_{ij}^{k-1} if d_{ij}^k doesn't use node k

$d_{ik}^{k-1} + d_{kj}^{k-1}$ if d_{ij}^k does use node k

Formalizing Subproblems

$u \to v$ path P: "intermediate nodes" are all nodes in P other than u, v.

d_{ij}^k: distance from i to j using only $i \to j$ paths with intermediate vertices in $\{1, 2, \ldots, k\}$.

- Goal: compute d_{ij}^k for all $i, j, k \in [n]$.
- Return d_{ij}^n for all $i, j \in V$.

$$d_{ij}^k = \begin{cases} \ell(i,j) & \text{if } k = 0 \\ \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) & \text{if } k \geq 1 \end{cases}$$

Optimal Substructure

Theorem

For all $i, j, k \in [n]$*:*

$$d_{ij}^{k} = \begin{cases} \ell(i,j) & \text{if } k = 0 \\ \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) & \text{if } k \geq 1 \end{cases}$$

Optimal Substructure

Theorem

For all $i, j, k \in [n]$*:*

$$d_{ij}^{k} = \begin{cases} \ell(i,j) & \text{if } k = 0 \\ \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) & \text{if } k \geq 1 \end{cases}$$

If $k = 0$: ✓

Optimal Substructure

Theorem

For all $i, j, k \in [n]$:

$$d_{ij}^{k} = \begin{cases} \ell(i,j) & \text{if } k = 0 \\ \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) & \text{if } k \geq 1 \end{cases}$$

If $k = 0$: ✓

If $k \geq 1$: prove $\leq$ and $\geq$

Optimal Substructure

Theorem

For all $i, j, k \in [n]$*:*

$$d_{ij}^{k} = \begin{cases} \ell(i,j) & \text{if } k = 0 \\ \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) & \text{if } k \geq 1 \end{cases}$$

If $k = 0$: ✓

If $k \geq 1$: prove $\leq$ and $\geq$

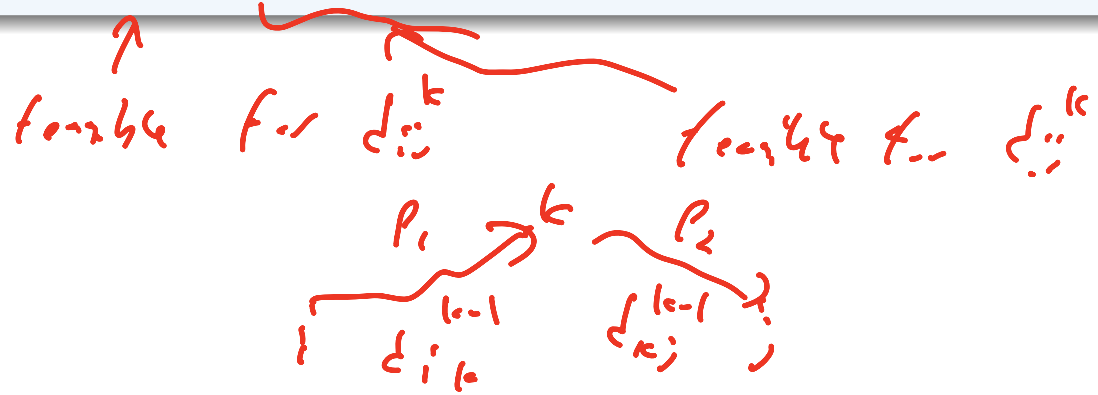
$\leq$:

Optimal Substructure

Theorem

For all $i, j, k \in [n]$:

$$d_{ij}^{k} = \begin{cases} \ell(i,j) & \text{if } k = 0 \\ \min\left(d_{ij}^{k-1},\ d_{ik}^{k-1} + d_{kj}^{k-1}\right) & \text{if } k \geq 1 \end{cases}$$



If $k = 0$: ✓

If $k \geq 1$: prove $\leq$ and $\geq$

$\leq$: Two feasible solutions

Optimal Substructure

Theorem

For all $i, j, k \in [n]$:

$$d_{ij}^k = \begin{cases} \ell(i,j) & \text{if } k = 0 \\ \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) & \text{if } k \geq 1 \end{cases}$$

If $k = 0$: ✓

If $k \geq 1$: prove $\leq$ and $\geq$

$\leq$: Two feasible solutions

$\geq$: Let P be shortest $i \to j$ path with all intermediate nodes in $[k]$

- If k not an intermediate node of P:

Optimal Substructure

Theorem

For all $i, j, k \in [n]$:

$$d_{ij}^k = \begin{cases} \ell(i,j) & \text{if } k = 0 \\ \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) & \text{if } k \geq 1 \end{cases}$$

If $k = 0$: ✓

If $k \geq 1$: prove $\leq$ and $\geq$

$\leq$: Two feasible solutions

$\geq$: Let P be shortest $i \to j$ path with all intermediate nodes in $[k]$

- If k not an intermediate node of P: P has all intermediate nodes in $[k-1]$ $\implies$ $\min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) \leq d_{ij}^{k-1} \leq \ell(P) = d_{ij}^k$

Optimal Substructure

Theorem

For all $i, j, k \in [n]$*:*

$$d_{ij}^k = \begin{cases} \ell(i,j) & \textit{if } k = 0 \\ \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) & \textit{if } k \geq 1 \end{cases}$$

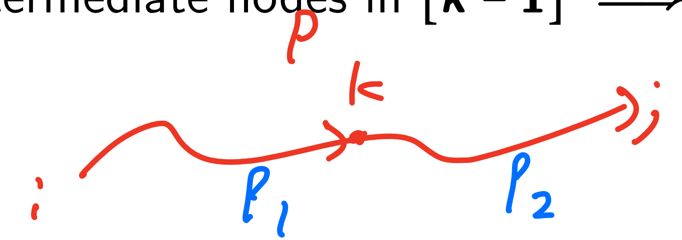
If $k = 0$: ✓

If $k \geq 1$: prove $\leq$ and $\geq$

$\leq$: Two feasible solutions

$\geq$: Let P be shortest $i \to j$ path with all intermediate nodes in $[k]$

- If k not an intermediate node of P: P has all intermediate nodes in $[k-1] \implies \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) \leq d_{ij}^{k-1} \leq \ell(P) = d_{ij}^k$
- If k is an intermediate node of P:



Optimal Substructure

Theorem

For all $i, j, k \in [n]$*:*

$$d_{ij}^k = \begin{cases} \ell(i,j) & \textit{if } k = 0 \\ \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) & \textit{if } k \geq 1 \end{cases}$$

If $k = 0$: ✓

If $k \geq 1$: prove $\leq$ and $\geq$

$\leq$: Two feasible solutions

$\geq$: Let P be shortest $i \to j$ path with all intermediate nodes in $[k]$

- If k not an intermediate node of P: P has all intermediate nodes in $[k-1]$ $\implies$ $\min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) \leq d_{ij}^{k-1} \leq \ell(P) = d_{ij}^k$
- If k is an intermediate node of P: divide P into P_1 (subpath from i to k) and P_2 (subpath from k to j)

$$\min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) \leq d_{ik}^{k-1} + d_{kj}^{k-1} \leq \ell(P_1) + \ell(P_2) = \ell(P) = d_{ij}^k$$

Floyd-Warshall Algorithm

Usually bottom-up, since so simple:

$M[i,j,0] = \ell(i,j)$ for all $i,j \in [n]$
for($k = 1$ to n)
    for($i = 1$ to n)
        for($j = 1$ to n)
            $M[i,j,k] = \min(M[i,j,k-1], M[i,k,k-1] + M[k,j,k-1])$

Floyd-Warshall Algorithm

Usually bottom-up, since so simple:

$M[i,j,0] = \ell(i,j)$ for all $i,j \in [n]$
for($k = 1$ to n)
    for($i = 1$ to n)
        for($j = 1$ to n)
            $M[i,j,k] = \min(M[i,j,k-1], M[i,k,k-1] + M[k,j,k-1])$

Correctness: obvious for $k = 0$. For $k \geq 1$:

$$\begin{aligned} M[i,j,k] &= \min(M[i,j,k-1], M[i,k,k-1] + M[k,j,k-1]) && \text{(def of algorithm)} \\ &= \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) && \text{(induction)} \\ &= d_{ij}^{k} && \text{(optimal substructure)} \end{aligned}$$

Floyd-Warshall Algorithm

Usually bottom-up, since so simple:

$M[i,j,0] = \ell(i,j)$ for all $i,j \in [n]$
for($k = 1$ to n)
  for($i = 1$ to n)
    for($j = 1$ to n)
      $M[i,j,k] = \min(M[i,j,k-1], M[i,k,k-1] + M[k,j,k-1])$

Correctness: obvious for $k = 0$. For $k \geq 1$:

$$
\begin{aligned}
M[i,j,k] &= \min(M[i,j,k-1], M[i,k,k-1] + M[k,j,k-1]) && \text{(def of algorithm)} \\
&= \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) && \text{(induction)} \\
&= d_{ij}^{k} && \text{(optimal substructure)}
\end{aligned}
$$

Running Time: $O(n^3)$

Fun Fact

arXiv > cs > arXiv:1904.01210

Search...

Help | Adva

Computer Science > Data Structures and Algorithms

[Submitted on 2 Apr 2019]

Incorrect implementations of the Floyd--Warshall algorithm give correct solutions after three repeats

Ikumi Hide, Soh Kumabe, Takanori Maehara

The Floyd--Warshall algorithm is a well-known algorithm for the all-pairs shortest path problem that is simply implemented by triply nested loops. In this study, we show that the incorrect implementations of the Floyd--Warshall algorithm that misorder the triply nested loops give correct solutions if these are repeated three times.

Subjects: **Data Structures and Algorithms (cs.DS)**
Cite as: arXiv:1904.01210 **[cs.DS]**
(or arXiv:1904.01210v1 **[cs.DS]** for this version)
https://doi.org/10.48550/arXiv.1904.01210

Submission history

From: Takanori Maehara [view email]
[v1] Tue, 2 Apr 2019 04:39:28 UTC (4 KB)

Johnson's Algorithm

Reweighting

Different Approach: Can we "fix" negative weights so Dijkstra from every node works?

- Time would be $O(n(m + n \log n)) = O(mn + n^2 \log n)$, better than Floyd-Warshall

Reweighting

Different Approach: Can we "fix" negative weights so Dijkstra from every node works?

- Time would be $O(n(m + n \log n)) = O(mn + n^2 \log n)$, better than Floyd-Warshall

First attempt: Let $-\alpha$ be smallest length (most negative). Add α to every edge.

- Does this work?

Reweighting

Different Approach: Can we "fix" negative weights so Dijkstra from every node works?

- Time would be $O(n(m + n \log n)) = O(mn + n^2 \log n)$, better than Floyd-Warshall

First attempt: Let $-\alpha$ be smallest length (most negative). Add α to every edge.

- Does this work? No!

Reweighting

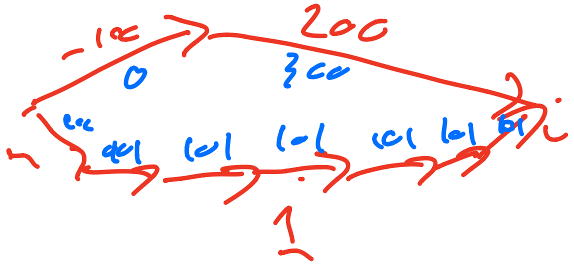
Different Approach: Can we "fix" negative weights so Dijkstra from every node works?

- Time would be $O(n(m + n \log n)) = O(mn + n^2 \log n)$, better than Floyd-Warshall

First attempt: Let $-\alpha$ be smallest length (most negative). Add α to every edge.

- Does this work? No!

- New length of path P is $\ell(P) + \alpha|P|$, so original shortest path might no longer be shortest path if it has many edges.



Reweighting

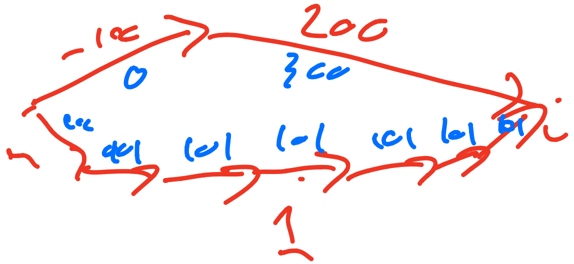
Different Approach: Can we "fix" negative weights so Dijkstra from every node works?

- Time would be $O(n(m + n\log n)) = O(mn + n^2 \log n)$, better than Floyd-Warshall

First attempt: Let $-\alpha$ be smallest length (most negative). Add α to every edge.

- Does this work? No!

- New length of path P is $\ell(P) + \alpha|P|$, so original shortest path might no longer be shortest path if it has many edges.



Some other kind of reweighting? Need new lengths $\hat{\ell}$ such that:

Reweighting

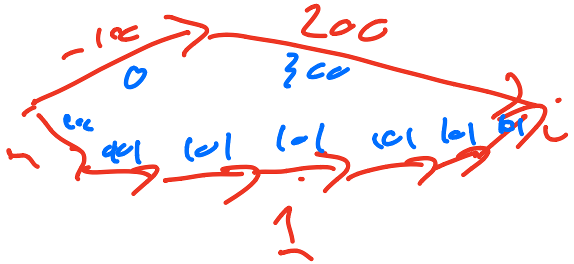
Different Approach: Can we "fix" negative weights so Dijkstra from every node works?

- Time would be $O(n(m + n \log n)) = O(mn + n^2 \log n)$, better than Floyd-Warshall

First attempt: Let $-\alpha$ be smallest length (most negative). Add α to every edge.

- Does this work? No!

- New length of path P is $\ell(P) + \alpha|P|$, so original shortest path might no longer be shortest path if it has many edges.



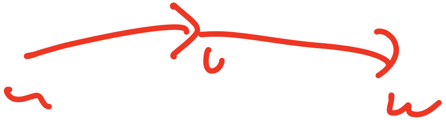
Some other kind of reweighting? Need new lengths $\hat{\ell}$ such that:

- Path P a shortest path under lengths ℓ if and only P a shortest path under lengths $\hat{\ell}$
- $\hat{\ell}(u, v) \geq 0$ for all $(u, v) \in E$

Vertex Reweighting

Neat observation: put weights at *vertices*!

- Let $h : V \to \mathbb{R}$ be node weights.
- Let $\ell_h(u, v) = \ell(u, v) + h(u) - h(v)$



Vertex Reweighting

Neat observation: put weights at *vertices*!

- Let $h : V \to \mathbb{R}$ be node weights.
- Let $\ell_h(u, v) = \ell(u, v) + h(u) - h(v)$

Let $P = \langle v_0, v_1, \ldots, v_k \rangle$ be arbitrary (not necessarily shortest) path.

Vertex Reweighting

Neat observation: put weights at *vertices*!

- Let $h : V \to \mathbb{R}$ be node weights.
- Let $\ell_h(u, v) = \ell(u, v) + h(u) - h(v)$

Let $P = \langle v_0, v_1, \ldots, v_k \rangle$ be arbitrary (not necessarily shortest) path.

$$\ell_h(P) = \sum_{i=0}^{k-1} \ell_h(v_i, v_{i+1}) = \sum_{i=0}^{k-1} (\ell(v_i, v_{i+1}) + h(v_i) - h(v_{i+1}))$$

Vertex Reweighting

Neat observation: put weights at *vertices*!

- Let $h : V \to \mathbb{R}$ be node weights.
- Let $\ell_h(u, v) = \ell(u, v) + h(u) - h(v)$

Let $P = \langle v_0, v_1, \ldots, v_k \rangle$ be arbitrary (not necessarily shortest) path.

$$\ell_h(P) = \sum_{i=0}^{k-1} \ell_h(v_i, v_{i+1}) = \sum_{i=0}^{k-1} \left(\ell(v_i, v_{i+1}) + h(v_i) - h(v_{i+1})\right)$$

$$= h(v_0) - h(v_k) + \sum_{i=0}^{k-1} \ell(v_i, v_{i+1}) \qquad \text{(telescoping)}$$

Vertex Reweighting

Neat observation: put weights at *vertices*!

- Let $h : V \to \mathbb{R}$ be node weights.
- Let $\ell_h(u, v) = \ell(u, v) + h(u) - h(v)$

Let $P = \langle v_0, v_1, \ldots, v_k \rangle$ be arbitrary (not necessarily shortest) path.

$$\ell_h(P) = \sum_{i=0}^{k-1} \ell_h(v_i, v_{i+1}) = \sum_{i=0}^{k-1} \left(\ell(v_i, v_{i+1}) + h(v_i) - h(v_{i+1})\right)$$

$$= h(v_0) - h(v_k) + \sum_{i=0}^{k-1} \ell(v_i, v_{i+1}) \qquad \text{(telescoping)}$$

$$= \ell(P) + h(v_0) - h(v_k)$$

Vertex Reweighting

Neat observation: put weights at *vertices*!

- Let $h : V \to \mathbb{R}$ be node weights.
- Let $\ell_h(u, v) = \ell(u, v) + h(u) - h(v)$

Let $P = \langle v_0, v_1, \ldots, v_k \rangle$ be arbitrary (not necessarily shortest) path.

$$\ell_h(P) = \sum_{i=0}^{k-1} \ell_h(v_i, v_{i+1}) = \sum_{i=0}^{k-1} (\ell(v_i, v_{i+1}) + h(v_i) - h(v_{i+1}))$$

$$= h(v_0) - h(v_k) + \sum_{i=0}^{k-1} \ell(v_i, v_{i+1}) \qquad \text{(telescoping)}$$

$$= \ell(P) + h(v_0) - h(v_k)$$

$h(v_0) - h(v_k)$ added to *every* $v_0 \to v_k$ path, so shortest path from v_0 to v_k still shortest path!

Making lengths nonnegative

So vertex reweighting preserves shortest paths. Find weights to make lengths nonnegative?

Add *new node* $\boldsymbol{s}$ to graph, edges $(\boldsymbol{s}, \boldsymbol{v})$ for all $\boldsymbol{v} \in \boldsymbol{V}$ of length $\mathbf{0}$

Making lengths nonnegative

So vertex reweighting preserves shortest paths. Find weights to make lengths nonnegative?

Add *new node* $\boldsymbol{s}$ to graph, edges $(\boldsymbol{s}, \boldsymbol{v})$ for all $\boldsymbol{v} \in \boldsymbol{V}$ of length $\mathbf{0}$

- Run Bellman-Ford from $\boldsymbol{s}$, then for all $\boldsymbol{u} \in \boldsymbol{V}$ set $\boldsymbol{h}(\boldsymbol{u})$ to be $\boldsymbol{d}(\boldsymbol{s}, \boldsymbol{u})$
- Note $\boldsymbol{h}(\boldsymbol{u}) \leq \mathbf{0}$ for all $\boldsymbol{u} \in \boldsymbol{V}$

Making lengths nonnegative

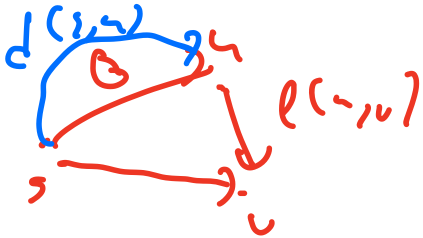
So vertex reweighting preserves shortest paths. Find weights to make lengths nonnegative?

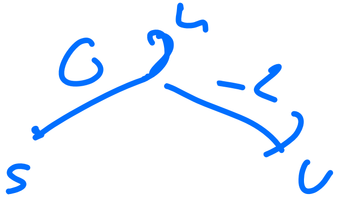
Add *new node* $\boldsymbol{s}$ to graph, edges $(\boldsymbol{s}, \boldsymbol{v})$ for all $\boldsymbol{v} \in \boldsymbol{V}$ of length $\mathbf{0}$

- Run Bellman-Ford from $\boldsymbol{s}$, then for all $\boldsymbol{u} \in \boldsymbol{V}$ set $\boldsymbol{h}(\boldsymbol{u})$ to be $\boldsymbol{d}(\boldsymbol{s}, \boldsymbol{u})$
- Note $\boldsymbol{h}(\boldsymbol{u}) \leq \mathbf{0}$ for all $\boldsymbol{u} \in \boldsymbol{V}$

Want to show that $\ell_{\boldsymbol{h}}(\boldsymbol{u}, \boldsymbol{v}) \geq \mathbf{0}$ for all edges $(\boldsymbol{u}, \boldsymbol{v})$.

- Triangle inequality: $\boldsymbol{h}(\boldsymbol{v}) = \boldsymbol{d}(\boldsymbol{s}, \boldsymbol{v}) \leq \boldsymbol{d}(\boldsymbol{s}, \boldsymbol{u}) + \ell(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{h}(\boldsymbol{u}) + \ell(\boldsymbol{u}, \boldsymbol{v})$





Making lengths nonnegative

So vertex reweighting preserves shortest paths. Find weights to make lengths nonnegative?

Add *new node* $\boldsymbol{s}$ to graph, edges $(\boldsymbol{s}, \boldsymbol{v})$ for all $\boldsymbol{v} \in \boldsymbol{V}$ of length $\mathbf{0}$

- Run Bellman-Ford from $\boldsymbol{s}$, then for all $\boldsymbol{u} \in \boldsymbol{V}$ set $\boldsymbol{h}(\boldsymbol{u})$ to be $\boldsymbol{d}(\boldsymbol{s}, \boldsymbol{u})$
- Note $\boldsymbol{h}(\boldsymbol{u}) \leq \mathbf{0}$ for all $\boldsymbol{u} \in \boldsymbol{V}$

Want to show that $\ell_{\boldsymbol{h}}(\boldsymbol{u}, \boldsymbol{v}) \geq \mathbf{0}$ for all edges $(\boldsymbol{u}, \boldsymbol{v})$.

- Triangle inequality: $\boldsymbol{h}(\boldsymbol{v}) = \boldsymbol{d}(\boldsymbol{s}, \boldsymbol{v}) \leq \boldsymbol{d}(\boldsymbol{s}, \boldsymbol{u}) + \ell(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{h}(\boldsymbol{u}) + \ell(\boldsymbol{u}, \boldsymbol{v})$

$$\ell_{\boldsymbol{h}}(\boldsymbol{u}, \boldsymbol{v}) = \ell(\boldsymbol{u}, \boldsymbol{v}) + \boldsymbol{h}(\boldsymbol{u}) - \boldsymbol{h}(\boldsymbol{v}) \geq \ell(\boldsymbol{u}, \boldsymbol{v}) + \boldsymbol{h}(\boldsymbol{u}) - (\boldsymbol{h}(\boldsymbol{u}) + \ell(\boldsymbol{u}, \boldsymbol{v})) = \mathbf{0}$$

def

Johnson's Algorithm

- Add vertex s to graph, edge (s, u) for all $u \in V$ with $\ell(s, u) = 0$ — $O(n)$
- Run Bellman-Ford from s, set $h(u) = d(s, u)$ — $O(mn)$
- Remove s, run Dijkstra from every node $u \in V$ using lengths ℓ_h to get $d_h(u, v)$ for all $u, v \in V$ — $O(n(m + n \log n))$
- If want distances, set $d(u, v) = d_h(u, v) - h(u) + h(v)$ for all $u, v \in V$ — $O(n^2)$

Correctness: From previous discussion.

Johnson's Algorithm

- Add vertex s to graph, edge (s, u) for all $u \in V$ with $\ell(s, u) = 0$
- Run Bellman-Ford from s, set $h(u) = d(s, u)$
- Remove s, run Dijkstra from every node $u \in V$ to get $d_h(u, v)$ for all $u, v \in V$
- If want distances, set $d(u, v) = d_h(u, v) - h(u) + h(v)$ for all $u, v \in V$

Correctness: From previous discussion.

Running Time:

Johnson's Algorithm

- Add vertex s to graph, edge (s, u) for all $u \in V$ with $\ell(s, u) = 0$
- Run Bellman-Ford from s, set $h(u) = d(s, u)$
- Remove s, run Dijkstra from every node $u \in V$ to get $d_h(u, v)$ for all $u, v \in V$
- If want distances, set $d(u, v) = d_h(u, v) - h(u) + h(v)$ for all $u, v \in V$

Correctness: From previous discussion.

Running Time: $O(n) + O(mn) + O(n(m + n \log n)) = O(mn + n^2 \log n)$