Lecture 16: All-Pairs Shortest Paths

Michael Dinitz

October 24, 2024

601.433/633 Introduction to Algorithms

Announcements

- Mid-Semester feedback on Courselore!
- No lecture notes

Introduction

Setup:

- Directed graph $G = (V, E)$
- Length $\ell(x, y)$ on each edge $(x, y) \in E$
- Length of path P is $\ell(P) = \sum_{(x,y)\in P} \ell(x, y)$
- $d(x, y) = \min_{x \to y \text{ paths } P} \ell(P)$

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Today: Distances between all pairs of nodes!

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Can we do better? Particularly for negative edge weights?

- Main goal today: Negative weights as fast as possible.

Floyd-Warshall Algorithm

Floyd-Warshall: A Different Dynamic Programming Approach

To simplify notation, let $V = \{1, 2, \ldots, n\}$ and $\ell(i,j) = \infty$ if $(i,j) \notin E$

Bellman-Ford subproblems: length of shortest path with at most some number of edges

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New subproblems:

- Intuition: "shortest path from u to v either goes through node n, or it doesn't"
  - If it doesn't: shortest uses only first nodes in $\{1, 2, \ldots, n-1\}$.
  - If it does: consists of a path P_1 from u to n and a path P_2 from n to v, neither of which uses n (internally).

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- Subproblems: shortest path from u to v that only uses nodes in $\{1, 2, \ldots k\}$ for all u, v, k.

Formalizing Subproblems

$u \to v$ path P: "intermediate nodes" are all nodes in P other than u, v.

d_{ij}^k: distance from i to j using only $i \to j$ paths with intermediate vertices in $\{1, 2, \ldots, k\}$.

- Goal: compute d_{ij}^k for all $i, j, k \in [n]$.
- Return d_{ij}^n for all $i, j \in V$.

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Optimal Substructure

Theorem

For all $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \in [\boldsymbol{n}]$*:*

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- If k is an intermediate node of P: divide P into P_1 (subpath from i to k) and P_2 (subpath from k to j)

$$\min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) \leq d_{ik}^{k-1} + d_{kj}^{k-1} \leq \ell(P_1) + \ell(P_2) = \ell(P) = d_{ij}^k$$

Floyd-Warshall Algorithm

Usually bottom-up, since so simple:

$M[i,j,0] = \ell(i,j)$ for all $i,j \in [n]$
for($k = 1$ to n)
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Running Time: $O(n^3)$

Fun Fact

arXiv > cs > arXiv:1904.01210

Search...

Help | Adva

Computer Science > Data Structures and Algorithms

[Submitted on 2 Apr 2019]

Incorrect implementations of the Floyd--Warshall algorithm give correct solutions after three repeats

Ikumi Hide, Soh Kumabe, Takanori Maehara

The Floyd--Warshall algorithm is a well-known algorithm for the all-pairs shortest path problem that is simply implemented by triply nested loops. In this study, we show that the incorrect implementations of the Floyd--Warshall algorithm that misorder the triply nested loops give correct solutions if these are repeated three times.

Subjects: **Data Structures and Algorithms (cs.DS)**
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(or arXiv:1904.01210v1 **[cs.DS]** for this version)
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From: Takanori Maehara [view email]
[v1] Tue, 2 Apr 2019 04:39:28 UTC (4 KB)

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Reweighting

Different Approach: Can we "fix" negative weights so Dijkstra from every node works?

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- Does this work?

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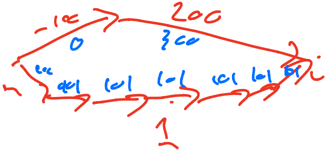

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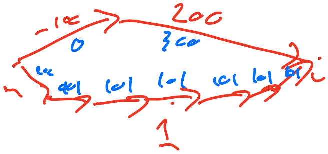

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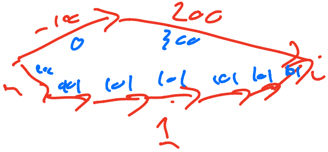

Some other kind of reweighting? Need new lengths $\hat{\ell}$ such that:

- Path P a shortest path under lengths ℓ if and only P a shortest path under lengths $\hat{\ell}$
- $\hat{\ell}(u, v) \geq 0$ for all $(u, v) \in E$

Vertex Reweighting

Neat observation: put weights at *vertices*!

- Let $h : V \to \mathbb{R}$ be node weights.
- Let $\ell_h(u, v) = \ell(u, v) + h(u) - h(v)$

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$$\ell_h(P) = \sum_{i=0}^{k-1} \ell_h(v_i, v_{i+1}) = \sum_{i=0}^{k-1} (\ell(v_i, v_{i+1}) + h(v_i) - h(v_{i+1}))$$

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$h(v_0) - h(v_k)$ added to *every* $v_0 \to v_k$ path, so shortest path from v_0 to v_k still shortest path!

Making lengths nonnegative

So vertex reweighting preserves shortest paths. Find weights to make lengths nonnegative?

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Want to show that $\ell_h(u, v) \geq 0$ for all edges (u, v).

- Triangle inequality: $h(v) = d(s, v) \leq d(s, u) + \ell(u, v) = h(u) + \ell(u, v)$

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$$\ell_h(u, v) = \ell(u, v) + h(u) - h(v) \geq \ell(u, v) + h(u) - (h(u) + \ell(u, v)) = 0$$

Johnson's Algorithm

- Add vertex s to graph, edge (s, u) for all $u \in V$ with $\ell(s, u) = 0$
- Run Bellman-Ford from s, set $h(u) = d(s, u)$
- Remove s, run Dijkstra from every node $u \in V$ to get $d_h(u, v)$ for all $u, v \in V$
- If want distances, set $d(u, v) = d_h(u, v) - h(u) + h(v)$ for all $u, v \in V$

Correctness: From previous discussion.

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Running Time: $O(n) + O(mn) + O(n(m + n \log n)) = O(mn + n^2 \log n)$