Lecture 17: Minimum Spanning Trees

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October 29, 2024
601.433/633 Introduction to Algorithms

Introduction

Definition

A *spanning tree* of an undirected graph $G = (V, E)$ is a set of edges $T \subseteq E$ such that (V, T) is connected and acyclic.

Definition

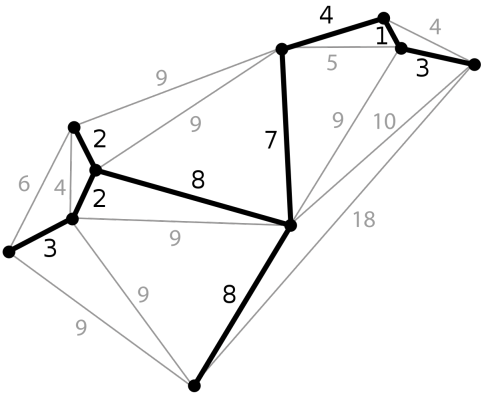
Minimum Spanning Tree problem (MST):

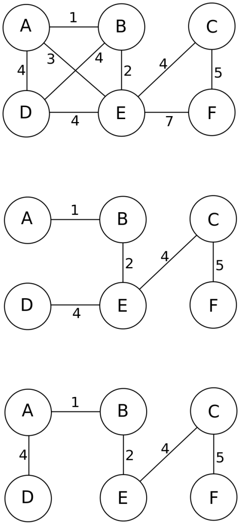
- Input:
  - Undirected graph $G = (V, E)$
  - Edge weights $w : E \to \mathbb{R}_{\geq 0}$
- Output: Spanning tree minimizing $w(T) = \sum_{e \in T} w(e)$.

Foundational problem in *network design*. Tons of applications.

Today: one "recipe", two different algorithms from recipe. Main idea: greedy.

Examples





Generic Algorithm

Generic Greedy

Definition

Suppose that A is subset of *some* MST. If $A \cup \{e\}$ is also a subset of some MST, then e is *safe* for A.

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Generic-MST {
    A = ∅
    while(A not a spanning tree) {
        find an edge e safe for A
        A = A ∪ {e}
    }
    return A
}
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Theorem

Generic-MST is correct: it always returns an MST.

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Claim: A always a subset of some MST.
Base case: ✓
Inductive step: ✓ □

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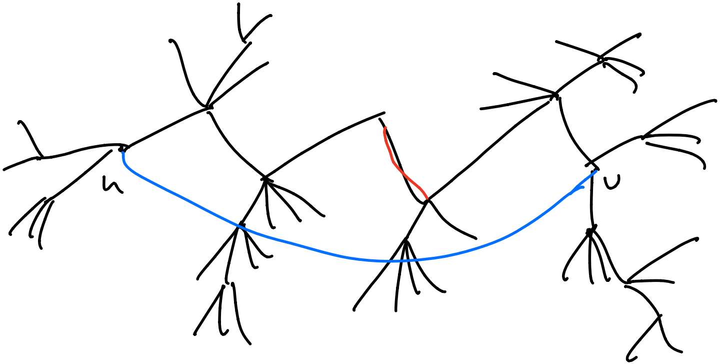
But how to find a safe edge? And which one to add?

Structural Properties

Lemma

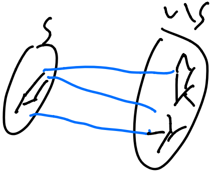
Let $\boldsymbol{T}$ be a spanning tree, let $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$, and let $\boldsymbol{P}$ be the $\boldsymbol{u}-\boldsymbol{v}$ path in $\boldsymbol{T}$. If $\{\boldsymbol{u}, \boldsymbol{v}\} \notin \boldsymbol{T}$, then $\boldsymbol{T}' = (\boldsymbol{T} \cup \{\{\boldsymbol{u}, \boldsymbol{v}\}\}) \setminus \{\boldsymbol{e}\}$ is a spanning tree for all $\boldsymbol{e} \in \boldsymbol{P}$.



Structural Properties

Definition

A *cut* $(S, V \setminus S)$ (or $(S, \bar{S})$ or just S) is a partition of V into two parts. Edge e *crosses* cut $(S, \bar{S})$ if e has one endpoint in S and one endpoint in $\bar{S}$.



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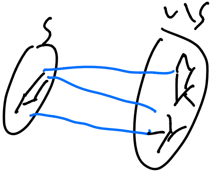

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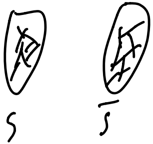
Cut $(S, \bar{S})$ *respects* $A \subseteq E$ if no edge in A crosses $(S, \bar{S})$



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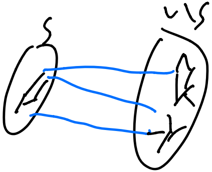

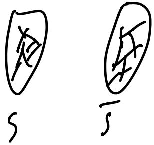

Definition

e is a *light edge* for $(S, \bar{S})$ if e crosses $(S, \bar{S})$ and $w(e) = \min_{e' \text{ crossing } (S,\bar{S})} w(e')$

Main Structural Theorem

Theorem

Let $\boldsymbol{A} \subseteq \boldsymbol{E}$ *be a subset of some MST* $\boldsymbol{T}$*, let* $(\boldsymbol{S}, \bar{\boldsymbol{S}})$ *be a cut respecting* $\boldsymbol{A}$*, and let* $\boldsymbol{e} = \{\boldsymbol{u}, \boldsymbol{v}\}$ *be a light edge for* $(\boldsymbol{S}, \bar{\boldsymbol{S}})$*. Then* $\boldsymbol{e}$ *is safe for* $\boldsymbol{A}$*.*

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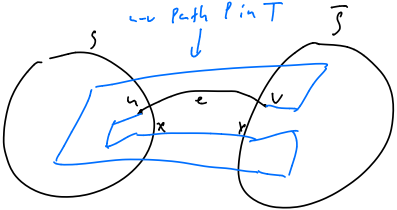
Let $T' = (T \cup \{e\}) \setminus \{\{x, y\}\}$
$\Longrightarrow$ T' a spanning tree by first lemma



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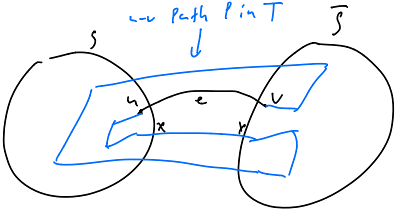

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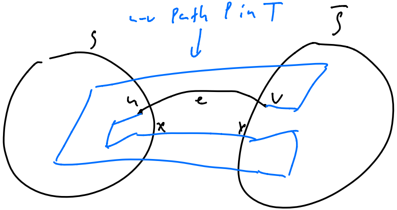
$$w(T') = w(T) + w(e) - w(x, y) \leq w(T)$$



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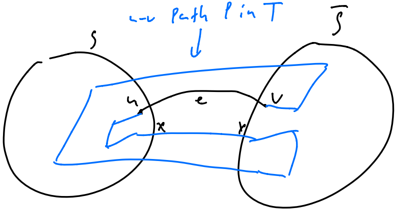

Prim's Algorithm

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Idea: start at arbitrary node $\boldsymbol{u}$. Greedily grow MST out of $\boldsymbol{u}$.

$\boldsymbol{A} = \varnothing$
Let $\boldsymbol{u}$ be an arbitrary node, and let $\boldsymbol{S} = \{\boldsymbol{u}\}$
while($\boldsymbol{A}$ is not a spanning tree) {
    Find an edge $\{\boldsymbol{x}, \boldsymbol{y}\}$ with $\boldsymbol{x} \in \boldsymbol{S}$ and $\boldsymbol{y} \notin \boldsymbol{S}$ that is light for $(\boldsymbol{S}, \bar{\boldsymbol{S}})$
    $\boldsymbol{A} \leftarrow \boldsymbol{A} \cup \{\{\boldsymbol{x}, \boldsymbol{y}\}\}$
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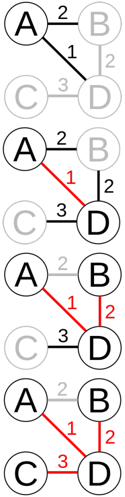

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Just Generic-MST!

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Just Generic-MST!

- $(S, \bar{S})$ always respects A (induction).
- If edge e added then light for $(S, \bar{S})$
- Hence e safe for A by main structural theorem.

□

Running Time

Trivial analysis:

- Every spanning tree has $n-1$ edges $\implies$ $n-1$ iterations
- In each iteration, look through all edges to find min-weight edge crossing $(S,\bar{S})$ $\implies$ $O(m)$ time
- Total $O(mn)$

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- When new vertex y added to S, need to update keys of nodes adjacent to y
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- n Inserts, n Extract-Mins, m Decrease-Keys
- Like Dijkstra, $O(m\log n)$ using binary heap. $O(m+n\log n)$ with Fibonacci heap (only Extract-Min is logarithmic)

Kruskal's Algorithm

Algorithm

Intuition: can we be *even greedier* than Prim's Algorithm?

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$\boldsymbol{A} = \varnothing$
Sort edges by weight (small to large)
For each edge $\boldsymbol{e}$ in this order $\{$
    if $\boldsymbol{A} \cup \{\boldsymbol{e}\}$ has no cycles, $\boldsymbol{A} = \boldsymbol{A} \cup \{\boldsymbol{e}\}$
$\}$
return $\boldsymbol{A}$

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Theorem

Kruskal's algorithm computes an MST.

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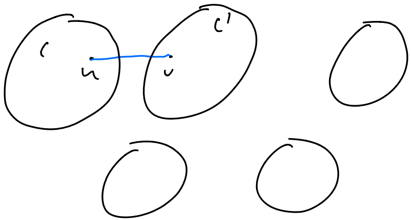

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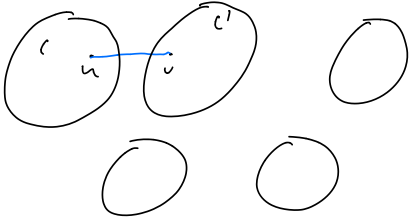

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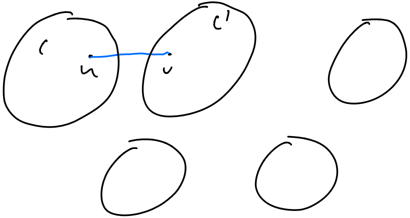

Consider cut $(C, \bar{C})$. Respects A, and $\{u, v\}$ light for it.
Main structural theorem $\implies$ $\{u, v\}$ safe for A

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Sorting dominates! $O(m \log n)$ total.