Lecture 18: Matroids and the Greedy Algorithm

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601.433/633 Introduction to Algorithms

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Question: when does greedy algorithm return optimal solution?

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- Universe U
- Collection $\mathcal{I} \subseteq 2^U$ (so $I \subseteq U$ for all $I \in \mathcal{I}$). Called *independent sets*
- Weights $w : U \to \mathbb{R}^+$

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Problem: find *max weight* independent set

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MST: weighted graph $G = (V, E, w)$. Find MST.

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For any tree T:

$$w'(T) = \sum_{e \in T} w'(e) = \sum_{e \in T} (\bar{w} - w(e)) = \sum_{e \in T} \bar{w} - \sum_{e \in T} w(e) = (n-1)\bar{w} - w(T)$$

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So under weights w', max-weight IS $=$ max-weight forest $=$ max-weight spanning tree $=$ min-weight spanning tree (weights w)

- So finding max-weight forest $=$ finding min spanning tree.

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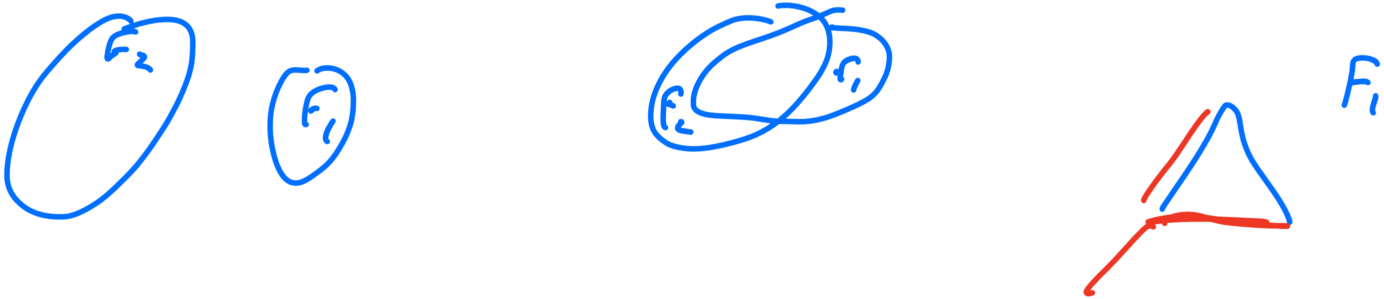

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Proof Sketch that Forests have Augmentation Property.

Suppose false: no edge in $F_2 \setminus F_1$ can be added to F_1. Let c_1 = # components in F_1, c_2 = # components in F_2

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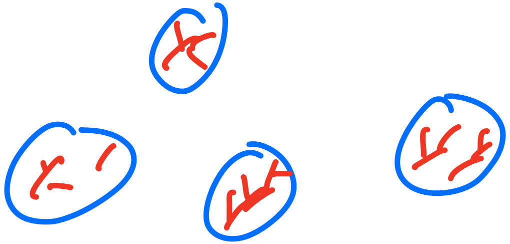

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Contradiction. □

Matroids

Definition

$(U, \mathcal{I})$ is a *matroid* if the following three properties hold:

1. $\varnothing \in \mathcal{I}$,
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$\varnothing$, a, $\{a,b\}$, c

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Warmup: In any matroid, the maximal independent sets (called bases) have the same size (called the rank of the matroid).

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Matroids: generalize both graph theory and linear algebra!

- Originally invented by Whitney as an attempt to generalize the concept of "linear independence"

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We'll assume we have independence oracle.

Greedy Algorithm

Kruskal, generalized to matroids (and max weight)!

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$F = \emptyset$
Sort U by weight (largest to smallest)
For each $u \in U$ in sorted order $\{$
    If $F \cup \{u\} \in \mathcal{I}$, add u to F
$\}$
Return F

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Theorem

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- $F = \{f_1, f_2, \ldots, f_r\}$, where $w(f_i) \geq w(f_{i+1})$ for all i (order added by greedy)
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Contradiction! Greedy would add e_z next, not f_j.

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So for hereditary set systems, matroids exactly characterize when the greedy algorithm works!

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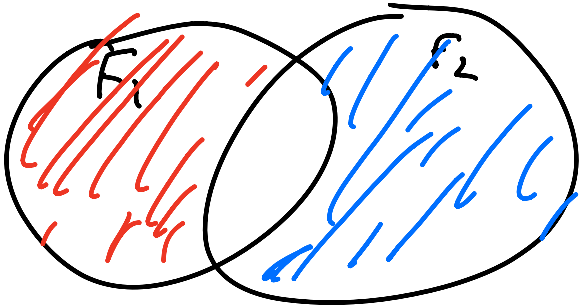

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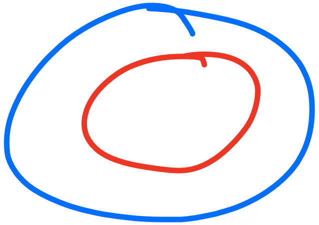
2. $|\boldsymbol{F_2} \setminus \boldsymbol{F_1}| \geq \mathbf{1}$
3. $|\boldsymbol{F_1} \setminus \boldsymbol{F_2}| \geq \mathbf{1}$ (hereditary)



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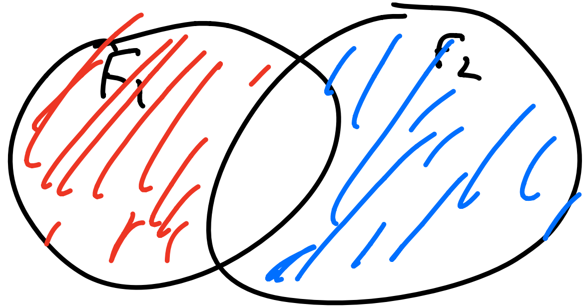

Easy facts:

1. $|F_2 \setminus F_1| > |F_1 \setminus F_2|$
2. $|F_2 \setminus F_1| \geq 2$
3. $|F_1 \setminus F_2| \geq 1$ (hereditary)

$\implies \exists \epsilon > 0$ such that $0 < (1 + \epsilon)|F_1 \setminus F_2| < |F_2 \setminus F_1|$

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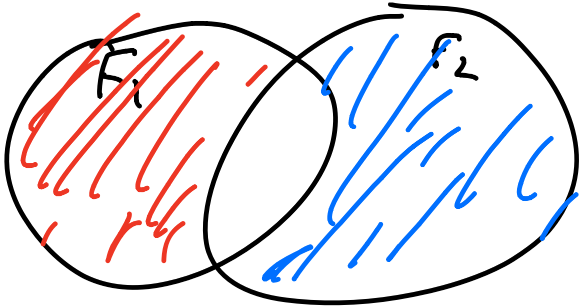
Contradiction. Suppose false $\implies (U, \mathcal{I})$ hereditary but not matroid.

$\implies \exists F_1, F_2 \in \mathcal{I}$ such that $|F_1| < |F_2|$ but $F_1 \cup \{e\} \notin \mathcal{I}$ for all $e \in F_2 \setminus F_1$



Easy facts:

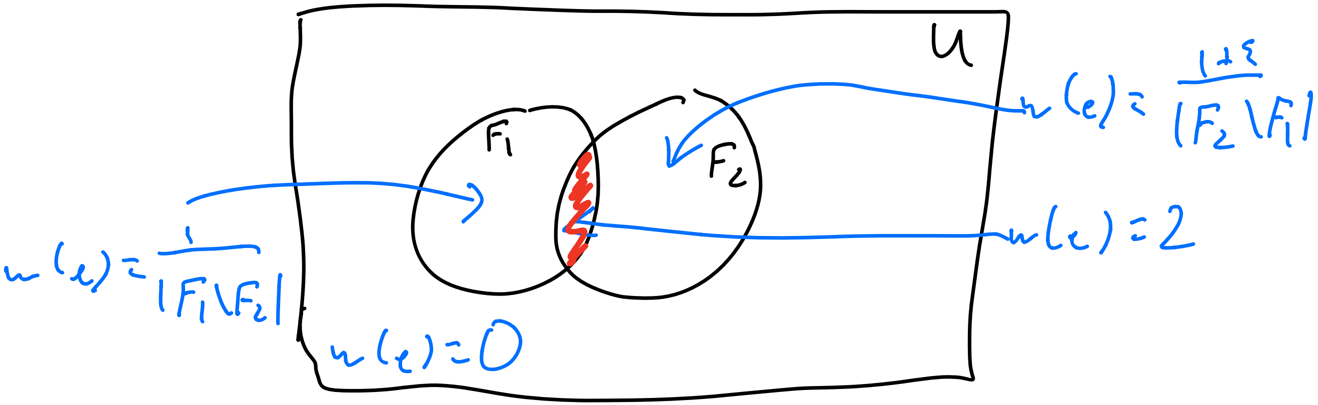
1. $|F_2 \setminus F_1| > |F_1 \setminus F_2|$
2. $|F_2 \setminus F_1| \geq 1$
3. $|F_1 \setminus F_2| \geq 1$ (hereditary)

$\implies \exists \epsilon > 0$ such that $0 < (1+\epsilon)|F_1 \setminus F_2| < |F_2 \setminus F_1|$

$$\implies \frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$$

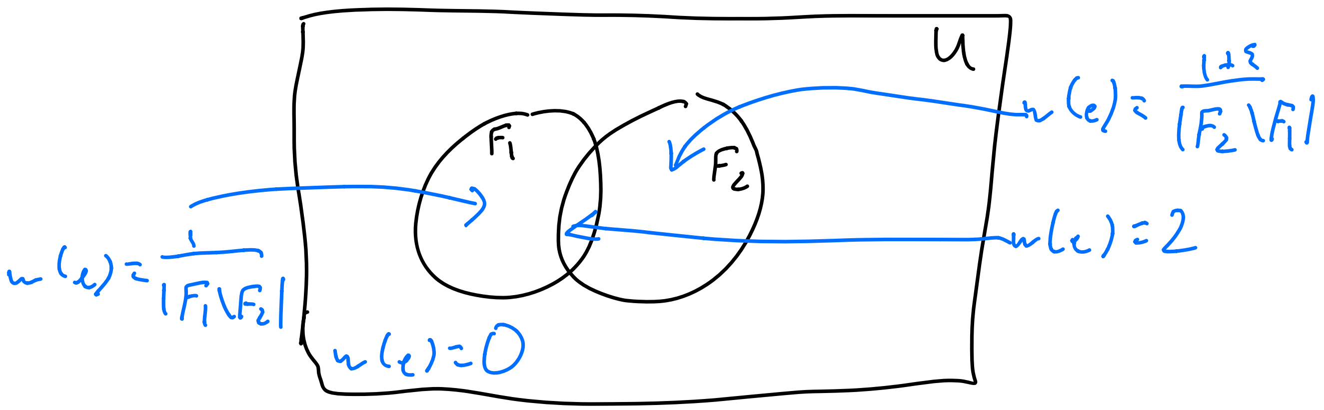
Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.

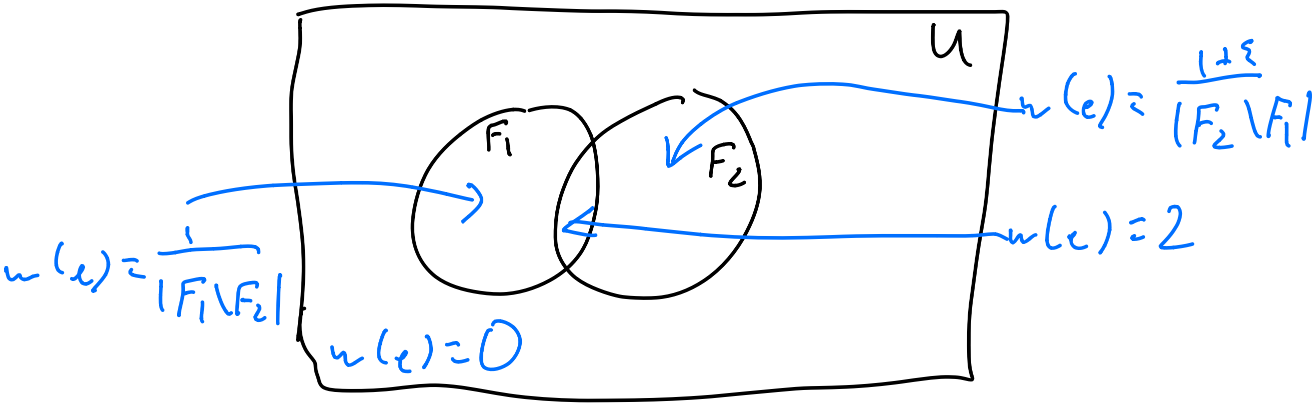


Greedy:

- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



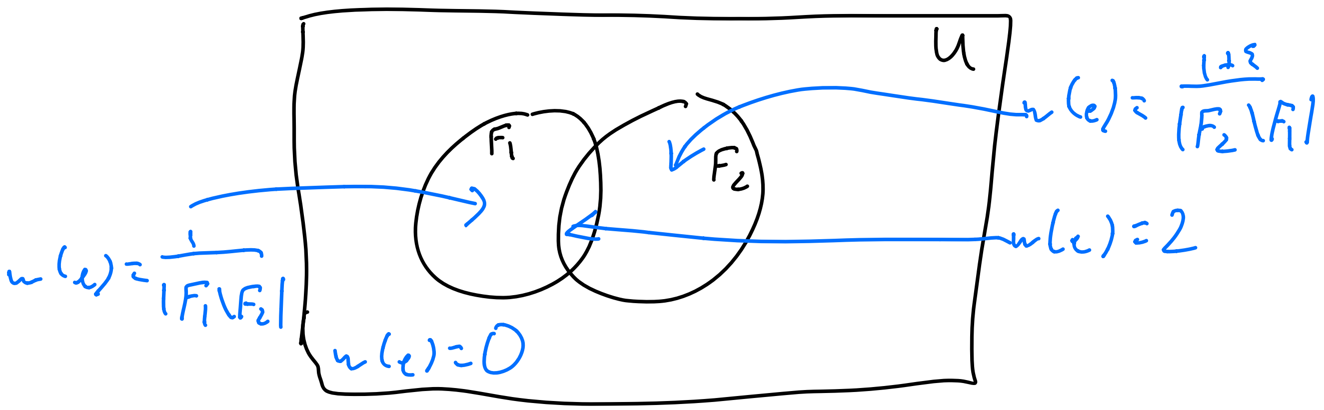
Greedy:

- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$
\begin{aligned}
w(\text{greedy}) &= 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|} \\
&= 2|F_1 \cap F_2| + 1
\end{aligned}
$$

Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



Greedy:

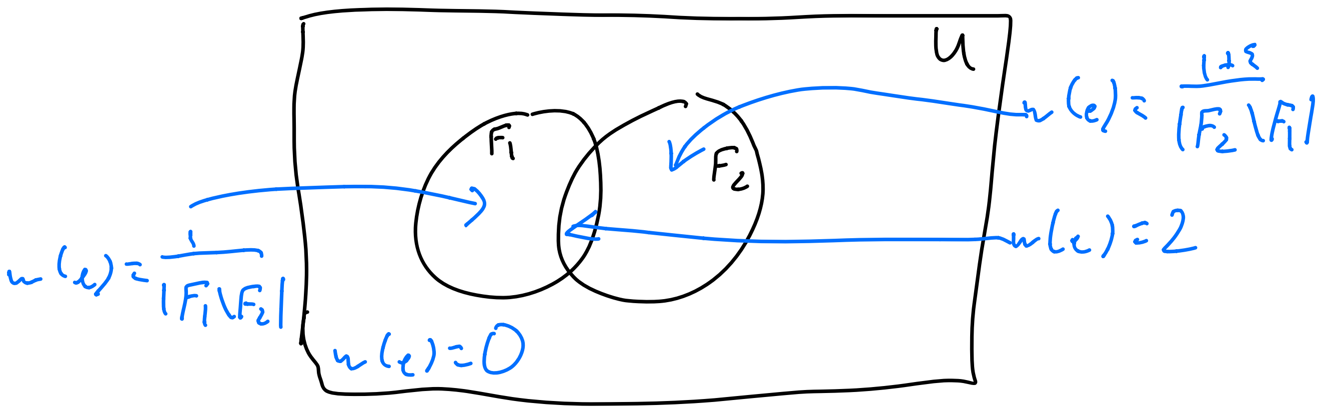
- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$w(\text{greedy}) = 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|} = 2|F_1 \cap F_2| + 1$$

$$w(F_2) = 2|F_1 \cap F_2| + |F_2 \setminus F_1| \frac{1+\epsilon}{|F_2 \setminus F_1|} = 2|F_1 \cap F_2| + 1 + \epsilon$$

Proof (cont'd)

Use fact that $\frac{1}{|F_1 \setminus F_2|} > \frac{1+\epsilon}{|F_2 \setminus F_1|}$ to define weights.



Greedy:

- Adds all of $F_1 \cap F_2$
- Adds all of $F_1 \setminus F_2$
- Can't add any of $F_2 \setminus F_1$

$$w(\text{greedy}) = 2|F_1 \cap F_2| + |F_1 \setminus F_2| \frac{1}{|F_1 \setminus F_2|} = 2|F_1 \cap F_2| + 1$$

$$w(F_2) = 2|F_1 \cap F_2| + |F_2 \setminus F_1| \frac{1+\epsilon}{|F_2 \setminus F_1|} = 2|F_1 \cap F_2| + 1 + \epsilon$$

Greedy not optimal: contradiction!