Lecture 19: Max-Flow Min-Cut

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November 5, 2024

601.433/633 Introduction to Algorithms

Introduction

Flow Network:

- Directed graph $G = (V, E)$
- *Capacities* $c : E \to \mathbb{R}_{\geq 0}$ (simplify notation: $c(x, y) = 0$ if $(x, y) \notin E$)
- Source $s \in V$, sink $t \in V$

Today: flows and cuts

- Flow: "sending stuff" from s to t
- Cut: separating t from s

Turn out to be very related!

Today: some algorithms but not efficient. Mostly structure. Better algorithms Thursday.

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Intuition: send "stuff" from s to t

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Value of flow $|f|$: "total amount of stuff sent from s to t"

$$|f| = \sum_{u:(s,u)\in E} f(s,u) - \sum_{u:(u,s)\in E} f(u,s) = \sum_{u:(u,t)\in E} f(u,t) - \sum_{u:(t,u)\in E} f(t,u)$$

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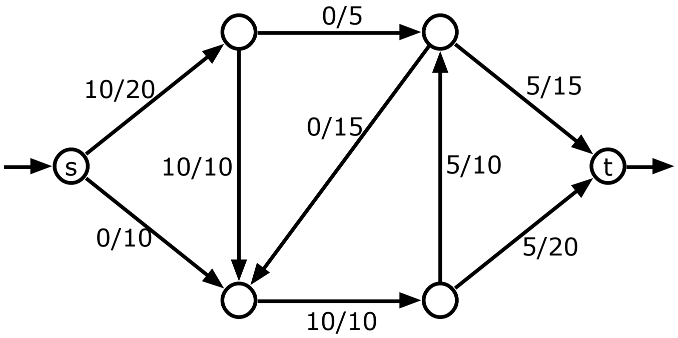

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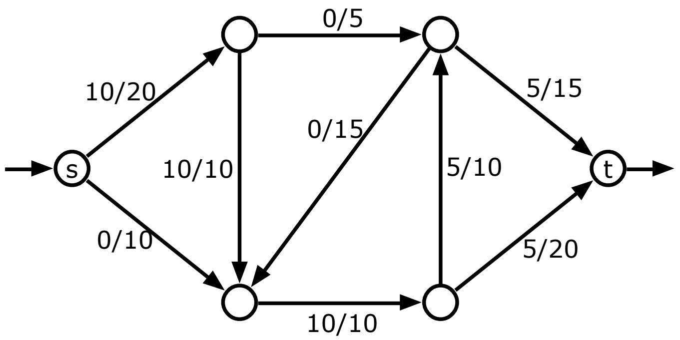

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Problem we'll talk about: find feasible flow of maximum value (max flow)

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$$cap(S, \bar{S}) = \sum_{(u,v) \in E : u \in S, v \in \bar{S}} c(u, v) = \sum_{u \in S} \sum_{v \in \bar{S}} c(u, v)$$

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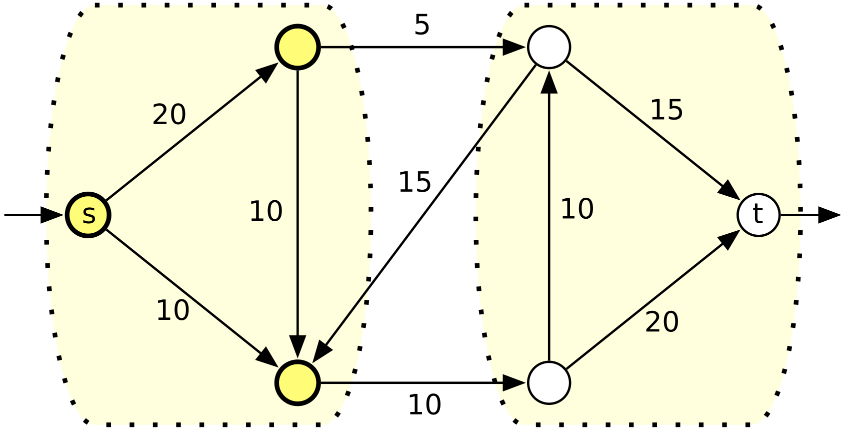

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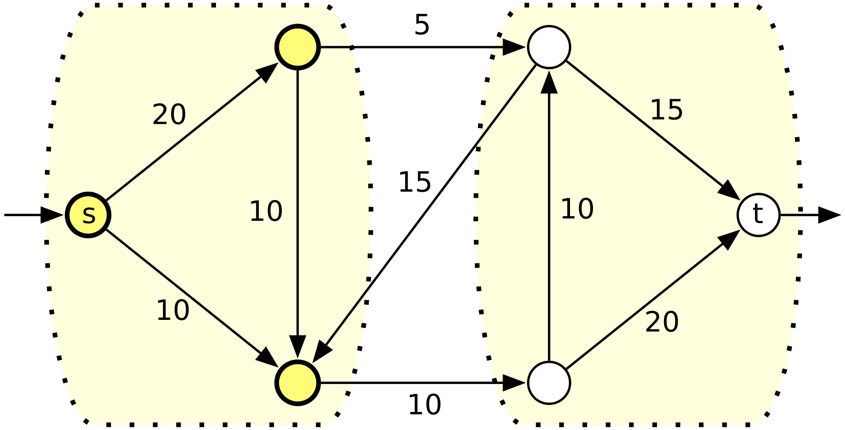
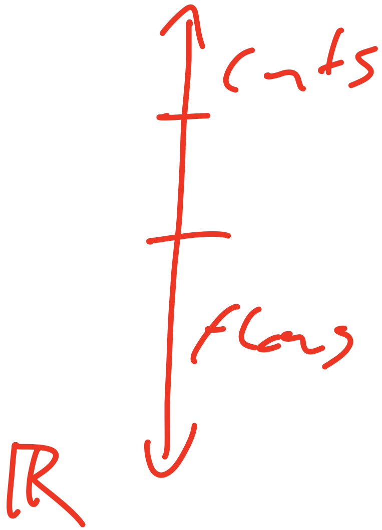


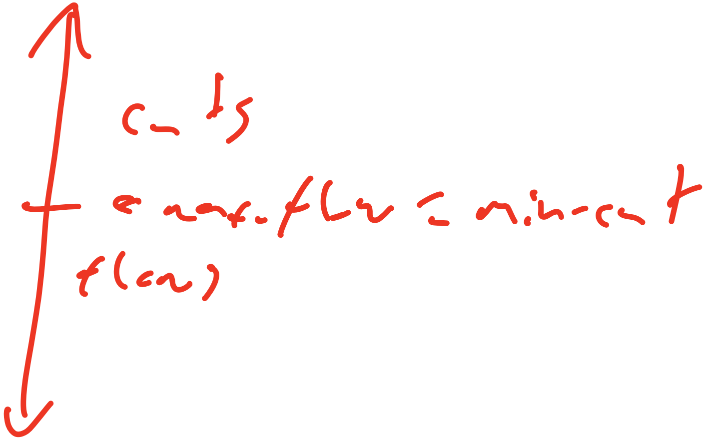
Problem we'll talk about: find (s,t)-cut of minimum capacity (min cut)

Warmup Theorem

Theorem

Let f be a feasible (s,t)-flow, and let $(S,\bar{S})$ be an (s,t)-cut. Then $|f| \le cap(S,\bar{S})$.





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$$= \sum_{u \in S} \left(\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) \right) \qquad \text{(flow conservation constraints)}$$

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&\le \sum_{u\in S}\sum_{v\in \bar{S}} f(u,v) && \text{(flow is nonnegative)}\\
&\le \sum_{u\in S}\sum_{v\in \bar{S}} c(u,v) = cap(S,\bar{S}) && \text{(flow is feasible)}
\end{aligned}$$

Max-Flow Min-Cut

Corollary

If $\boldsymbol{f}$ avoids every $\bar{\boldsymbol{S}} \to \boldsymbol{S}$ edge and saturates every $\boldsymbol{S} \to \bar{\boldsymbol{S}}$ edge, then $\boldsymbol{f}$ is a maximum flow and $(\boldsymbol{S}, \bar{\boldsymbol{S}})$ is a minimum cut.

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Theorem (Max-Flow Min-Cut Theorem)

In any flow network, value of max (s,t)-flow $=$ capacity of min (s,t)-cut.

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Theorem (Max-Flow Min-Cut Theorem)

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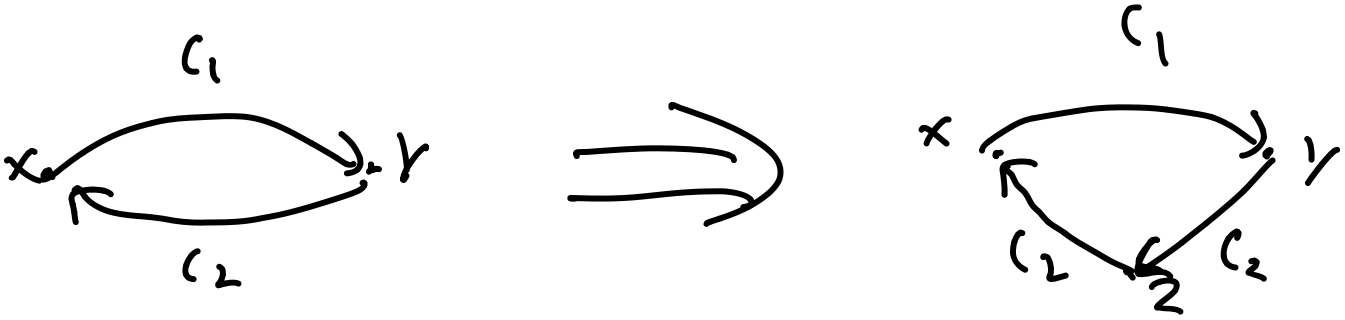
Spend rest of today proving this.

- Many different valid proofs.
- We'll see a classical proof which will naturally lead to algorithms for these problems.

One Direction

Cycles of length **2** will turn out to be annoying. Get rid of them.



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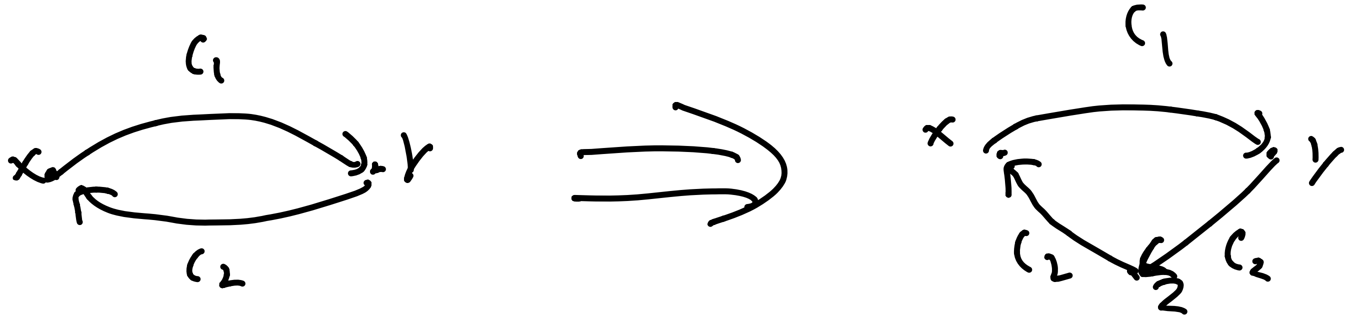

- Doesn't change max-flow or min-cut
- Increases #edges by constant factor, # nodes to original # edges.

Residual

Let f be feasible (s,t)-flow. Define *residual capacities*:

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

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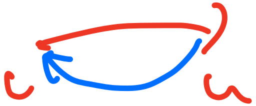

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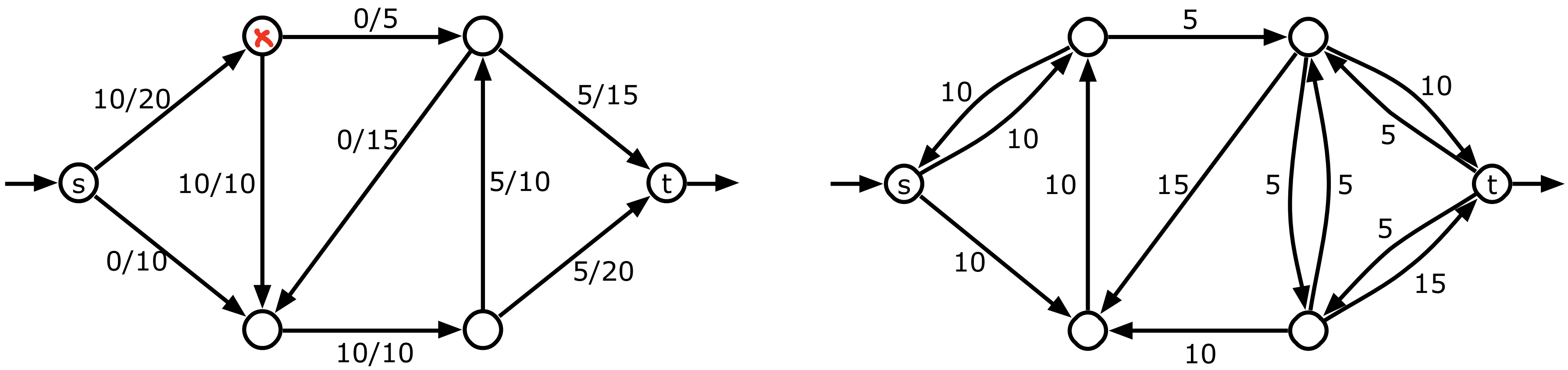
Residual Graph: $G_f = (V, E_f)$ where $(u,v) \in E_f$ if $c_f(u,v) > 0$.



A flow f in a weighted graph G and the corresponding residual graph G_f.

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Want to Show: There is a cut $(S, \bar{S})$ with $cap(S, \bar{S}) = |f|$.

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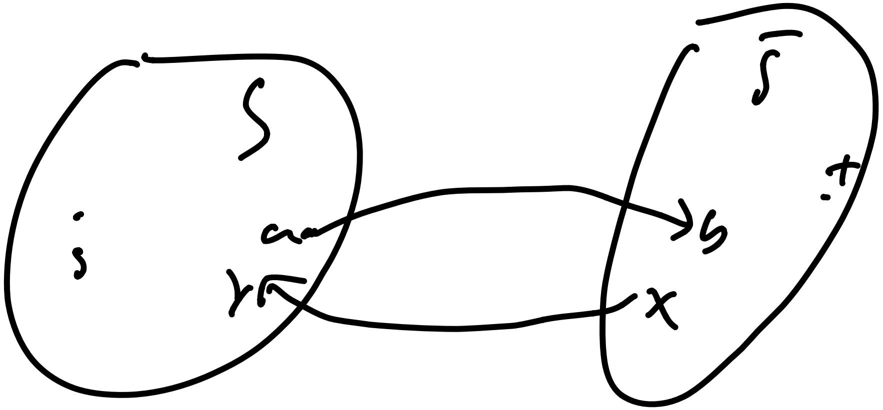

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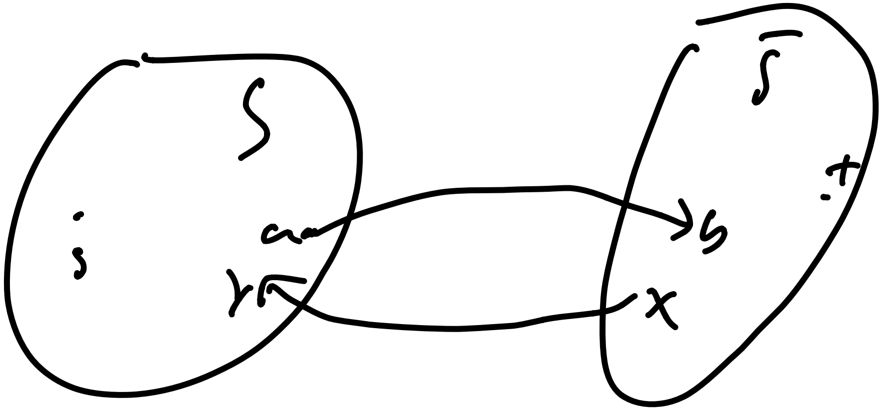

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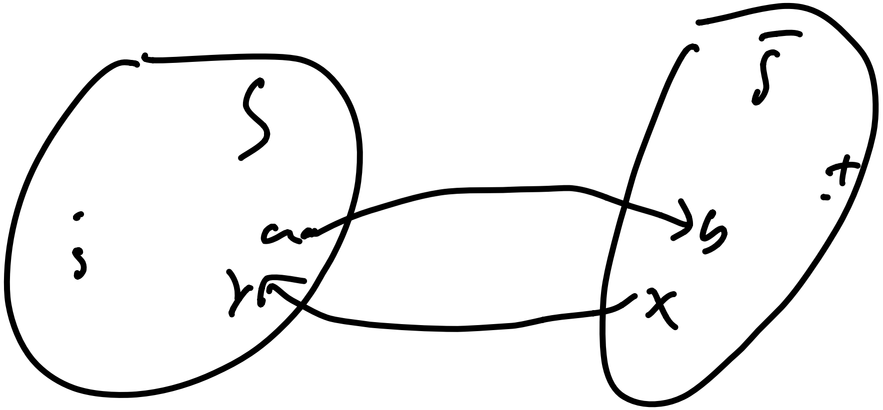

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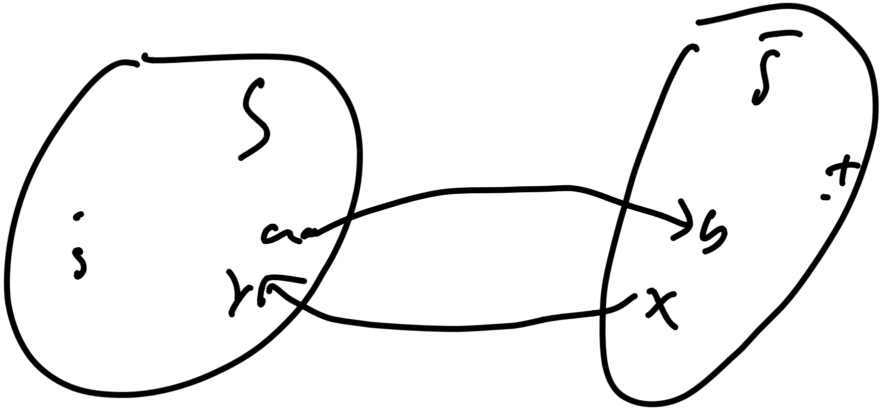

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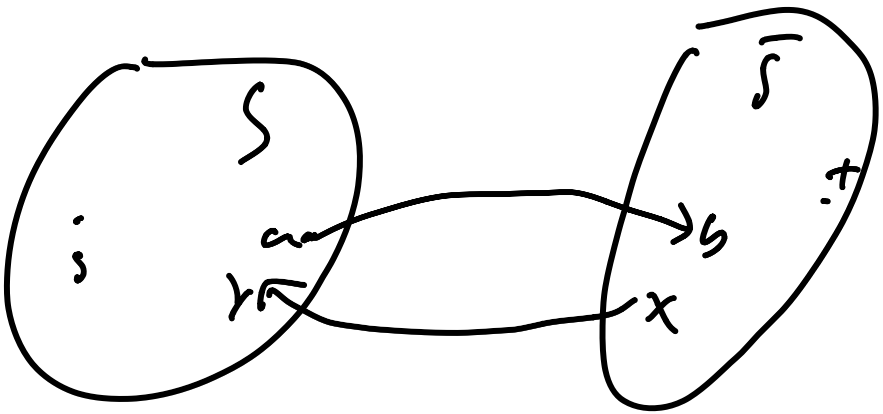

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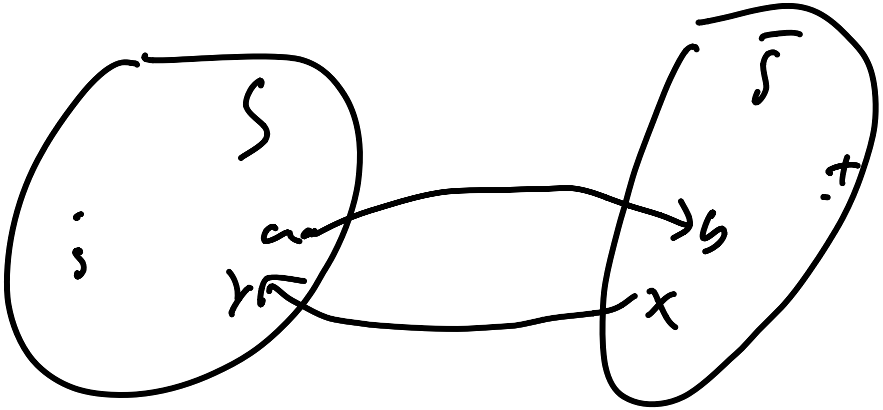

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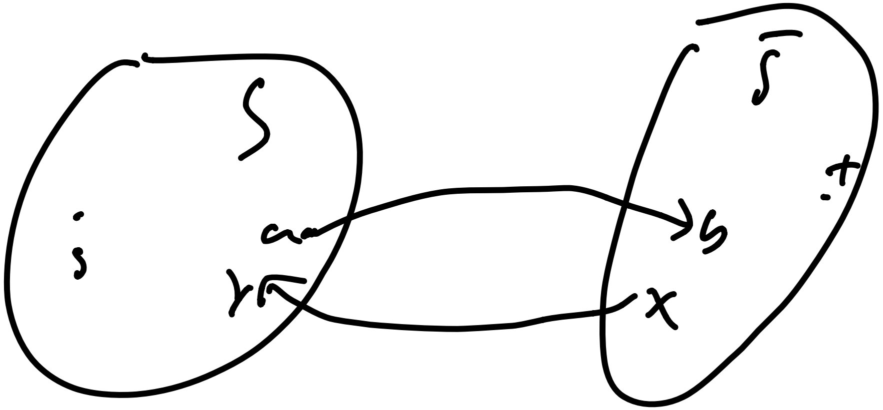

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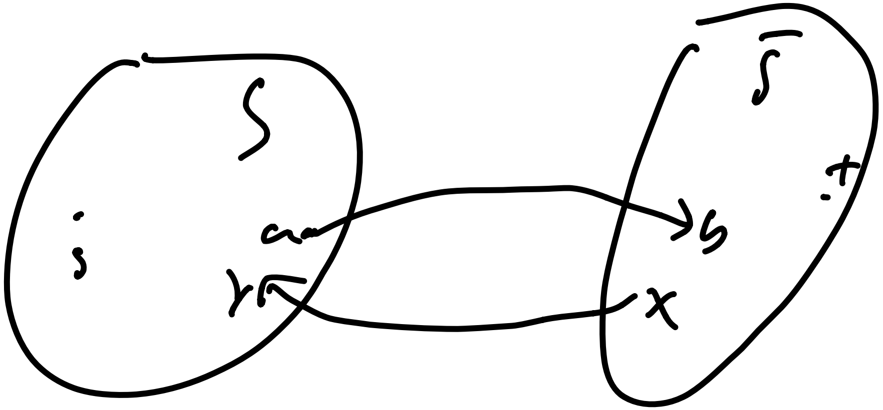

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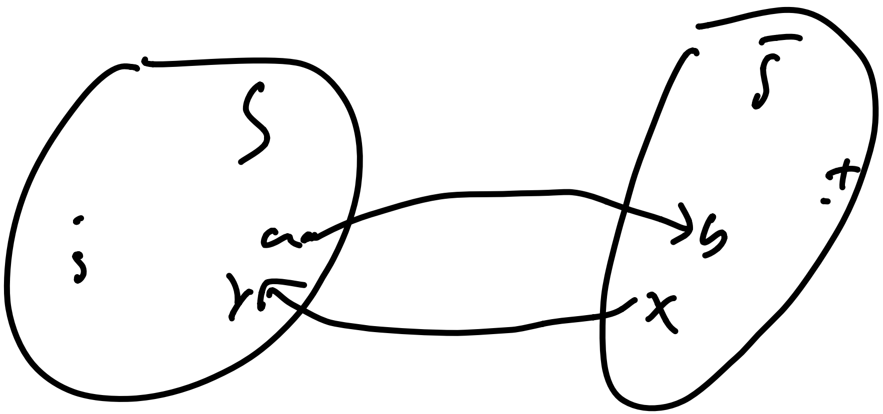

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f saturates $S \to \bar{S}$ edges, avoids $\bar{S} \to S$ edges $\implies cap(S, \bar{S}) = |f|$ by corollary

Case 2

Suppose $\exists$ an $s \to t$ path P in G_f.

- Called an *augmenting path*

Idea: show that we can "push" more flow along P, so f not a max flow. Contradiction, can't be in this case.

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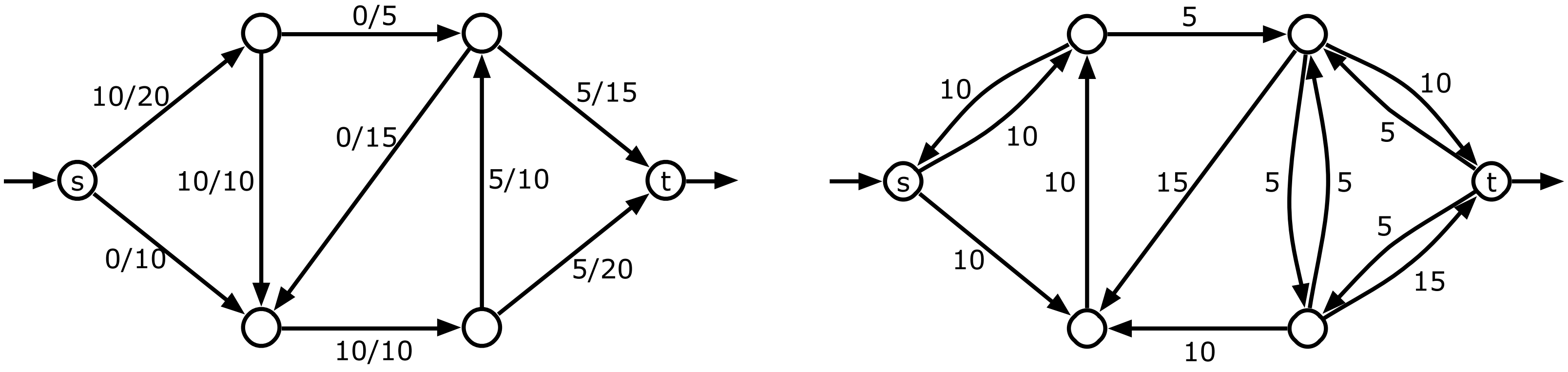
- Foreshadowing: augmenting path allows us to send more flow. Algorithm to increase flow!

Intuition



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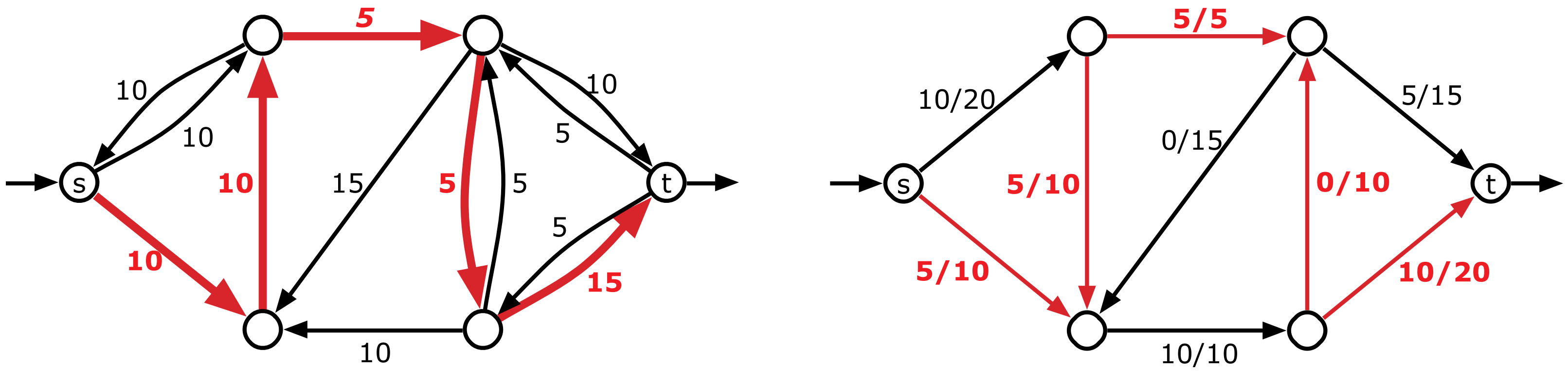

An augmenting path in G_f with value $F = 5$ and the augmented flow f'.

Formalities

Let P be (simple) augmenting path in G_f. Let $F = \min_{e \in P} c_f(e)$.

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Define new flow f': for all $(u, v) \in E$, let

$$f'(u,v) = \begin{cases} f(u,v) + F & \text{if } (u,v) \text{ in } P \\ f(u,v) - F & \text{if } (v,u) \text{ in } P \\ f(u,v) & \text{otherwise} \end{cases}$$

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Plan: prove (sketch) each subclaim individually

- $|f'| > |f|$
- f' an (s,t)-flow (flow conservation)
- f' feasible (obeys capacities)

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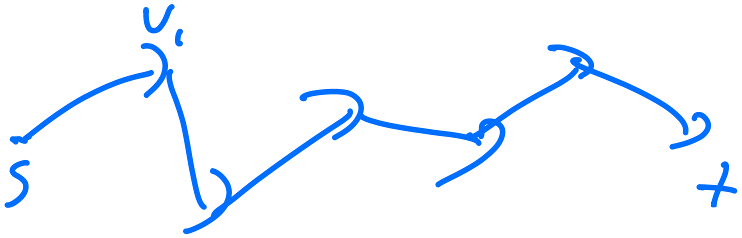

Consider first edge of P (out of s), say (s, v_1)

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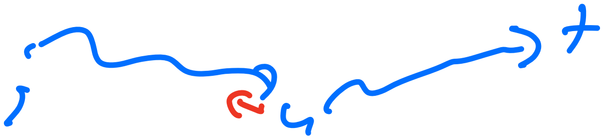

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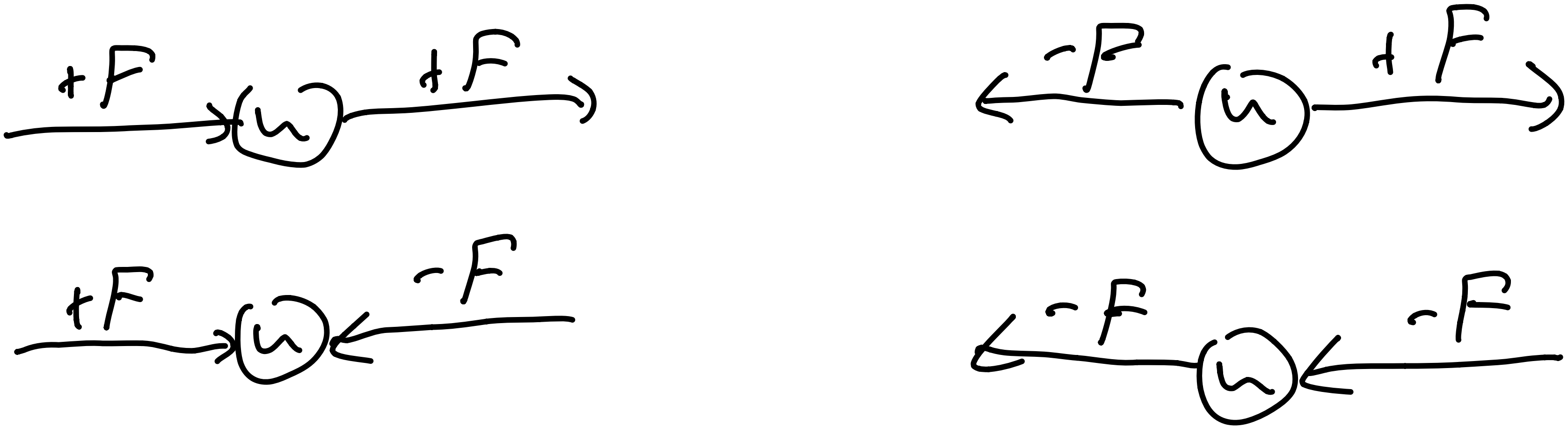

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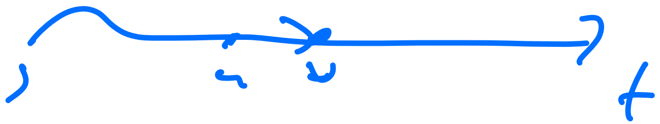

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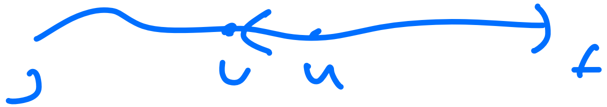

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Ford-Fulkerson Algorithm and Integrality

FF Algorithm

Obvious algorithm from previous proof: keep pushing flow!

$\boldsymbol{f} = \vec{\mathbf{0}}$
while($\exists \boldsymbol{s} \to \boldsymbol{t}$ path $\boldsymbol{P}$ in $\boldsymbol{G_f}$) {
    $\boldsymbol{F} = \mathbf{min}_{\boldsymbol{e} \in \boldsymbol{P}}\, \boldsymbol{c_f}(\boldsymbol{e})$
    Push $\boldsymbol{F}$ flow along $\boldsymbol{P}$ to get new flow $\boldsymbol{f}'$
    $\boldsymbol{f} = \boldsymbol{f}'$
}
return $\boldsymbol{f}$

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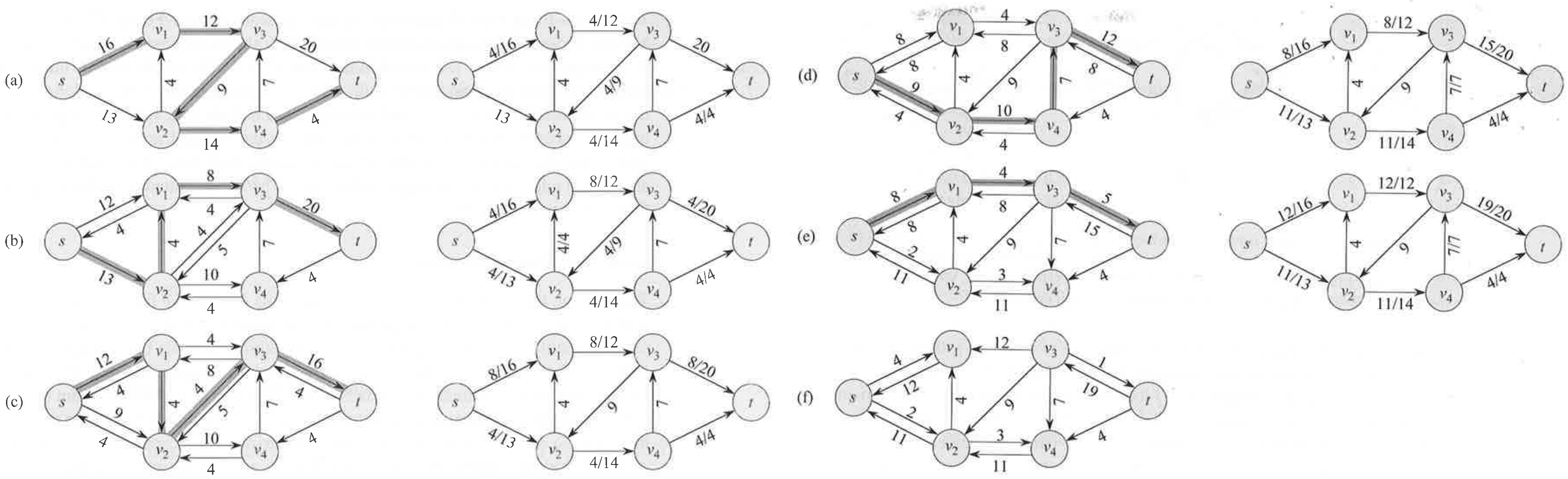
Correctness: directly from previous proof

Example



Integrality

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If all capacities are integers, then there is a max flow such that the flow through every edge is an integer

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Proof.

Induction on iterations of the Ford-Fulkerson algorithm: initially true, stays true $\implies$ true at end. □

Running Time

Theorem

If all capacities are integers and the max flow value is $\boldsymbol{F}$, Ford-Fulkerson takes time at most $\boldsymbol{O(F(m+n))}$

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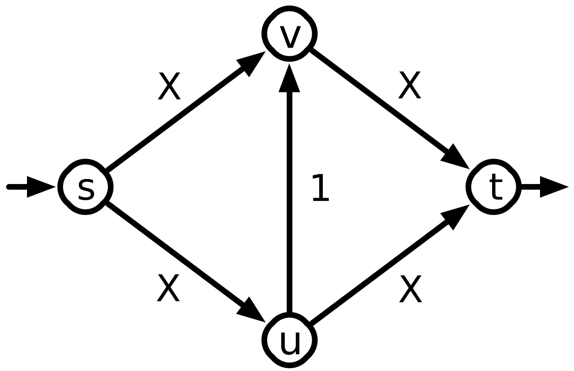
Finding path takes $\boldsymbol{O(m+n)}$ time, increase flow by at least $\mathbf{1}$



A bad example for the Ford-Fulkerson algorithm.

Running Time

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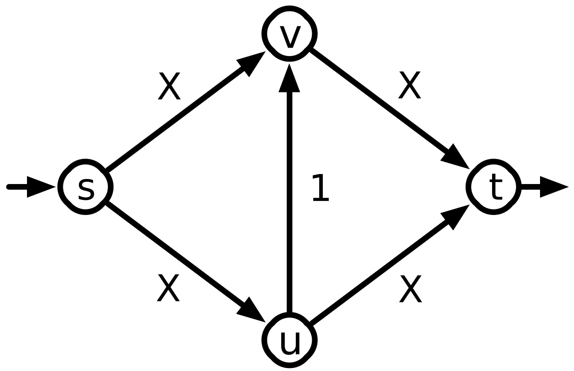

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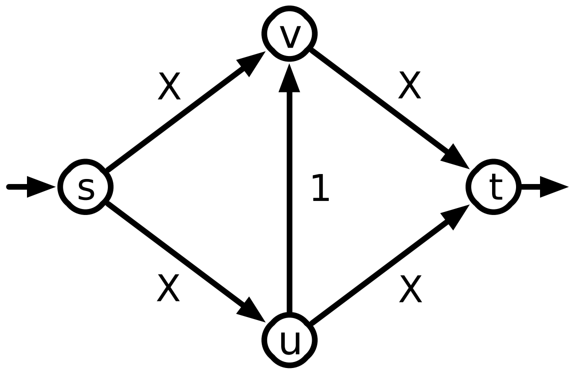

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- Running time: $\Omega(x)$

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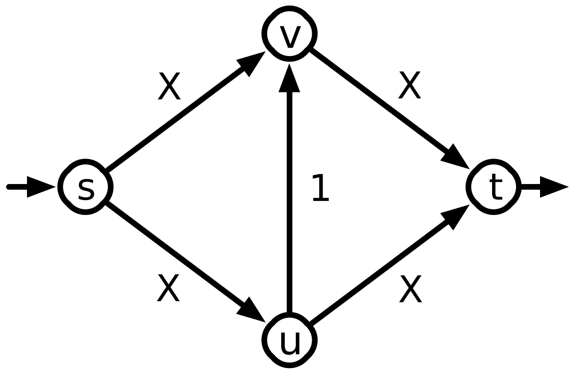

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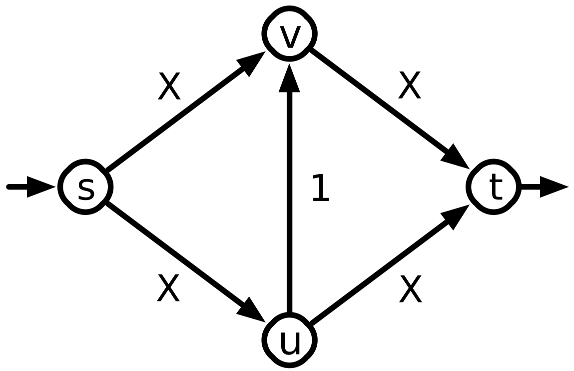

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Running time $\geq$ # iterations.

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$\implies$ Exponential time!