Lecture 2: Asymptotic Analysis, Recurrences

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August 29, 2024
601.433/633 Introduction to Algorithms

Things I Forget on Tuesday

Level of Formality:

- Part of mathematical maturity is knowing when to be formal, when not necessary
- Rule of thumb: Be formal for important parts
  - Problem 1 is *about* asymptotic notation. Be formal!
  - Problem 2 is *about* recurrences. Can be a little less formal with asymptotic notation.
- Lectures:
  - I tend to go fast, not be super formal. But I expect you to be formal in homeworks (unless stated otherwise)

Today

Should be review, some might be new.
See math background in CLRS

Asymptotics: $O(\cdot)$, $\Omega(\cdot)$, and $\Theta(\cdot)$ notation.

- Should know from Data Structures / MFCS. We'll be a bit more formal.
- Intuitively: hide constants and lower order terms, since we only care what happen "at scale" (asymptotically)

Recurrences: How to solve recurrence relations.

- Should know from MFCS / Discrete Math.

Asymptotic Notation

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Examples:

- $2n^2 + 27 = O(n^2)$: set $n_0 = 6$ and $c = 3$
- $2n^2 + 27 = O(n^3)$: same values, or $n_0 = 4$ and $c = 1$
- $n^3 + 2000n^2 + 2000n = O(n^3)$: set $n_0 = 10000$ and $c = 2$

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About *functions* not algorithms!
Expresses an *upper* bound

Example

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Theorem

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$\implies n^2 < 27$

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$\implies n^2 < 27 \implies n < 6$

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$\implies 2n^2 + 27 \leq 3n^2$ for all $n \geq 6$.

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Set $n_0 = 6$. Then $2n^2 + 27 \leq cn^2$ for all $n > n_0$. $\square$

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Set $n_0 = 6$. Then $2n^2 + 27 \leq cn^2$ for all $n > n_0$. $\square$

Many other ways to prove this!

$\Omega(\cdot)$

Counterpart to $O(\cdot)$: *lower* bound rather than upper bound.

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Examples:

- $2n^2 + 27 = \Omega(n^2)$: set $n_0 = 1$ and $c = 1$
- $2n^2 + 27 = \Omega(n)$: set $n_0 = 1$ and $c = 1$
- $\frac{1}{100}n^3 - 1000n^2 = \Omega(n^3)$: set $n_0 = 1000000$ and $c = 1/1000$

$\Theta(\cdot)$

Combination of $O(\cdot)$ and $\Omega(\cdot)$.

Definition

$g(n) \in \Theta(f(n))$ if $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$.

Note: constants n_0, c can be different in the proofs for $O(f(n))$ and $\Omega(f(n))$

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Equivalent:

Definition

$g(n) \in \Theta(f(n))$ if there are constants $c_1, c_2, n_0 > 0$ such that $c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all $n > n_0$.

Both lower bound and upper bound, so asymptotic equality.

Little notation

Strict versions of O and Ω:

Definition

$g(n) \in o(f(n))$ if for every constant $c > 0$ there exists a constant $n_0 > 0$ such that $g(n) < c \cdot f(n)$ for all $n > n_0$.

Definition

$g(n) \in \omega(f(n))$ if for every constant $c > 0$ there exists a constant $n_0 > 0$ such that $g(n) > c \cdot f(n)$ for all $n > n_0$.

Examples:

- $2n^2 + 27 = o(n^2 \log n)$
- $2n^2 + 27 = \omega(n)$

Recurrence Relations

Sorting

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  - Running time: Takes $O(n)$ time to find smallest unsorted element, decreases remaining unsorted by 1.
    $\implies T(n) = T(n-1) + cn$

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Also need base case. For algorithms, constant size input takes constant time.
$\implies T(n) \leq c$ for all $n \leq n_0$, for some constants $n_0, c > 0$.

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$$\begin{aligned} T(n) &= 3T(n/3) + n \leq 3(n/3)\log_3(3n/3) + n = n\log_3(n) + n \\ &= n(\log_3(n) + \log_3 3) = n\log_3(3n). \end{aligned}$$

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Example: selection sort. $T(n) = T(n-1) + cn$

Idea: "unroll" the recurrence.

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At least $n/2$ terms which are at least $cn/2 \implies T(n) \geq c\frac{n^2}{4} = \Omega(n^2)$

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Generalizes unrolling: draw out full tree of "recursive calls".

Mergesort: $T(n) = 2T(n/2) + cn$.

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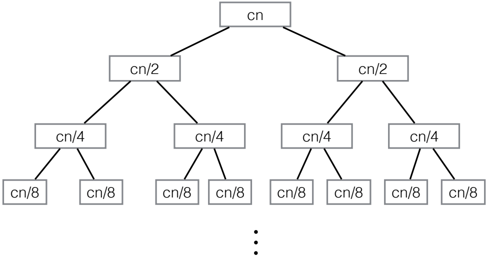

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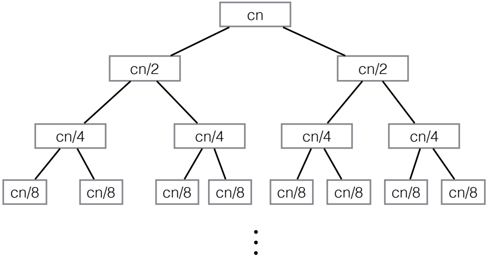

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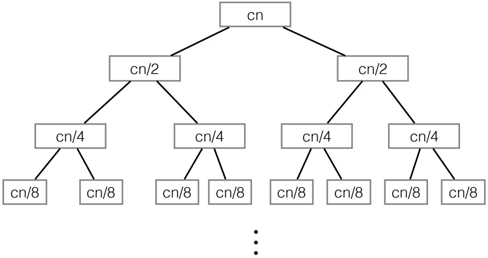

levels: $\log_2 n$

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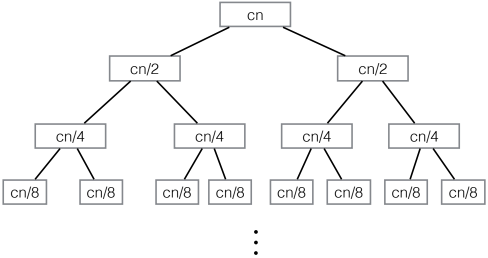

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Contribution of level i:

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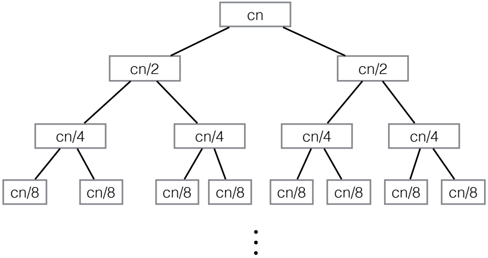

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Contribution of level i: $2^{i-1} cn/2^{i-1} = cn$

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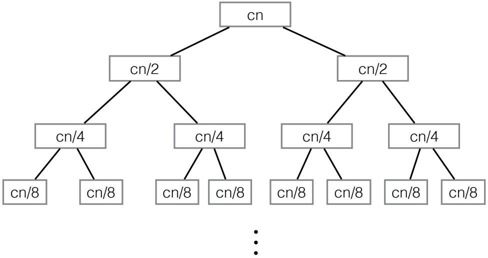

\# levels: $\log_2 n$

Contribution of level i: $2^{i-1}cn/2^{i-1} = cn$

$\implies T(n) = \Theta(n \log n)$

Recursion Tree: Strassen

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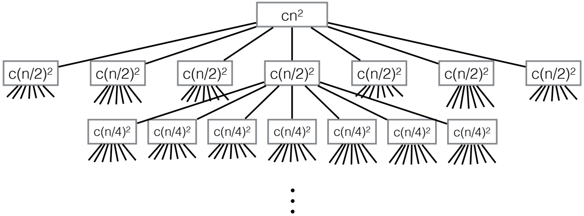

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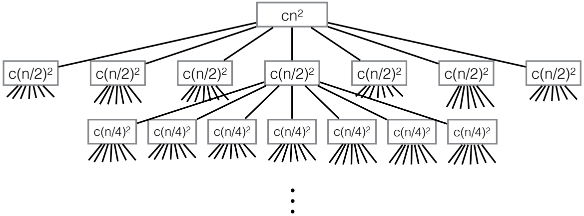

Level i: $7^{i-1}c(n/2^{i-1})^2 = (7/4)^{i-1}cn^2$

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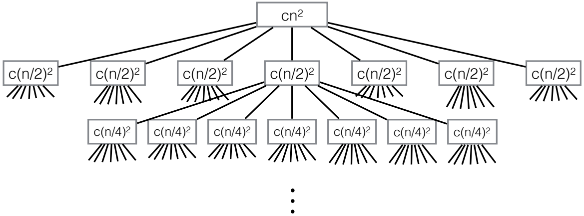

Level i: $7^{i-1}c(n/2^{i-1})^2 = (7/4)^{i-1}cn^2$

Total:
$$T(n) = \sum_{i=1}^{\log n+1}\left(\frac{7}{4}\right)^{i-1} cn^2 = cn^2\sum_{i=1}^{\log n+1}\left(\frac{7}{4}\right)^{i-1}$$

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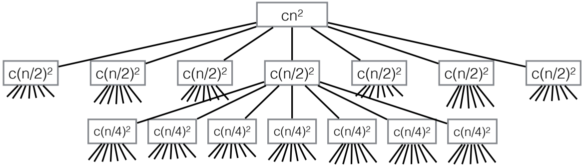

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$$\implies T(n) = O(n^2(7/4)^{\log n}) = O(n^2 n^{\log(7/4)}) = O(n^2 n^{\log 7-2})$$

$$= O(n^{\log 7})$$

Master Theorem

$$T(n) = aT(n/b) + cn^k \qquad\qquad T(1) = c$$

a, b, c, k constants with $a \geq 1$, $b > 1$, $c > 0$, and $k \geq 0$

Master Theorem

$$T(n) = aT(n/b) + cn^k \qquad T(1) = c$$

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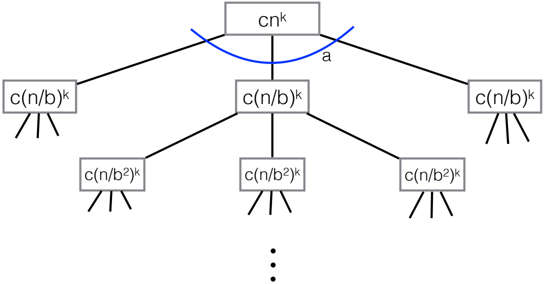

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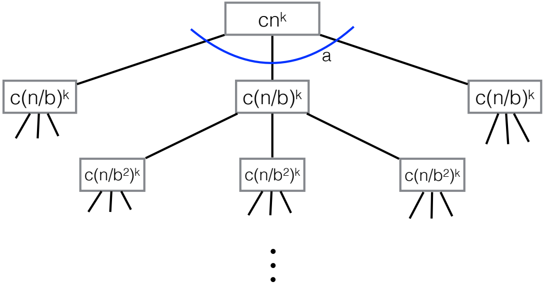

levels: $\log_b n + 1$

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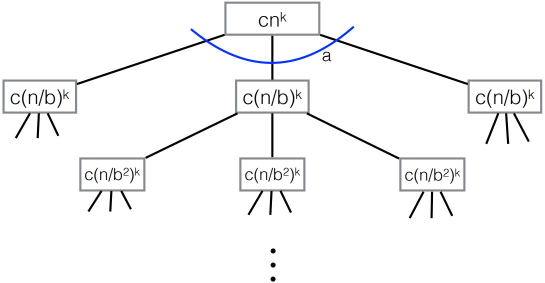

\# levels: $\log_b n + 1$

Level i: $a^{i-1} c(n/b^{i-1})^k = cn^k (a/b^k)^{i-1}$

Master Theorem II

Let $\alpha = (a/b^k)$

$$\implies T(n) = cn^k \sum_{i=1}^{\log_b n+1} (a/b^k)^{i-1} = cn^k \sum_{i=1}^{\log_b n+1} \alpha^{i-1}$$

Master Theorem II

Let $\alpha = (a/b^k)$

$\implies T(n) = cn^k \sum_{i=1}^{\log_b n+1} (a/b^k)^{i-1} = cn^k \sum_{i=1}^{\log_b n+1} \alpha^{i-1}$

- Case 1: $\alpha = 1$. All levels the same. $T(n) = cn^k \sum_{i=1}^{\log_b n+1} 1 = \Theta(n^k \log n)$

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Let $\alpha = (a/b^k)$

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- Case 2: $\alpha < 1$. Dominated by top level.

Master Theorem II

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  $\implies \sum_{i=1}^{\log_b n+1} \alpha^{i-1} \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \frac{1}{1-\alpha}$.
  $\implies T(n) = O(n^k)$

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  $T(n) \geq cn^k \implies T(n) = \Omega(n^k)$

Master Theorem II

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Master Theorem II

Let $\alpha = (a/b^k)$

$\implies T(n) = cn^k \sum_{i=1}^{\log_b n+1} (a/b^k)^{i-1} = cn^k \sum_{i=1}^{\log_b n+1} \alpha^{i-1}$

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- Case 3: $\alpha > 1$. Dominated by bottom level

Master Theorem II

Let $\alpha = (a/b^k)$

$\implies T(n) = cn^k \sum_{i=1}^{\log_b n+1} (a/b^k)^{i-1} = cn^k \sum_{i=1}^{\log_b n+1} \alpha^{i-1}$

- Case 1: $\alpha = 1$. All levels the same. $T(n) = cn^k \sum_{i=1}^{\log_b n+1} 1 = \Theta(n^k \log n)$
- Case 2: $\alpha < 1$. Dominated by top level.

  $\implies \sum_{i=1}^{\log_b n+1} \alpha^{i-1} \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \frac{1}{1-\alpha}$.

  $\implies T(n) = O(n^k)$

  $T(n) \geq cn^k \implies T(n) = \Omega(n^k) \implies T(n) = \Theta(n^k)$
- Case 3: $\alpha > 1$. Dominated by bottom level

$$\implies \sum_{i=1}^{\log_b n+1} \alpha^{i-1} = \alpha^{\log_b n} \sum_{i=1}^{\log_b n+1} \left(\frac{1}{\alpha}\right)^{i-1} \leq \alpha^{\log_b n} \frac{1}{1-(1/\alpha)}$$

$$= O(\alpha^{\log_b n})$$

Master Theorem II

Let $\alpha = (a/b^k)$

$\implies T(n) = cn^k \sum_{i=1}^{\log_b n+1} (a/b^k)^{i-1} = cn^k \sum_{i=1}^{\log_b n+1} \alpha^{i-1}$

- Case 1: $\alpha = 1$. All levels the same. $T(n) = cn^k \sum_{i=1}^{\log_b n+1} 1 = \Theta(n^k \log n)$
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$$= O(\alpha^{\log_b n})$$

$$\implies T(n) = \Theta(n^k \alpha^{\log_b n}) = \Theta(n^k (a/b^k)^{\log_b n}) = \Theta(a^{\log_b n})$$

$$= \Theta(n^{\log_b a})$$

Master Theorem III

Theorem ("Master Theorem")

The recurrence

$$T(n) = aT(n/b) + cn^k \qquad T(1) = c$$

where a, b, c, *and* k *are constants with* $a \geq 1$, $b > 1$, $c > 0$, *and* $k \geq 0$, *is equal to*

$$T(n) = \Theta(n^k) \text{ if } a < b^k,$$
$$T(n) = \Theta(n^k \log n) \text{ if } a = b^k,$$
$$T(n) = \Theta(n^{\log_b a}) \text{ if } a > b^k.$$