Lecture 21: Linear Programming

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601.433/633 Introduction to Algorithms

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Next semester: 601.438/638 Algorithmic Foundations of Differential Privacy

- Lots of fun algorithms!
- Very laid back: 3-4 homeworks, final project of your choosing

Differential Privacy: modern formal notion of privacy

- Super important in both practice and theory
- Used in US Census!
- I spent sabbatical at Google NYC working on privacy: fun theory, and really deployed!
- Allows us to think of privacy as a "resource" that we can analyze like other resources

Introduction

Today: What, why, and juste a taste of how

- Entire course on linear programming over in AMS. Super important topic!
- Fast algorithms in theory and in practice.

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Why: Even more general than max-flow, can still be solved in polynomial time!

- Max flow important in its own right, but also because it can be used to solve many other things (max bipartite matching)
- Linear programming: important in its own right, but also even more general than max-flow.
- Can model many, many problems!

Example: Planning Your Week (pre-COVID)

168 hours in a week. How much time to spend:

- Studying (S)
- Partying (P)
- Everything else (E)

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Constraints:

- $E \geq 56$ (at least 8 hours/day sleep, shower, etc.)
- $P + E \geq 70$ (need to stay sane)
- $S \geq 60$ (to pass your classes)
- $2S + E - 3P \geq 150$ (too much partying requires studying or sleep)

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Question: Suppose "happiness" is $2P + 3E$. Can we find a feasible solution maximizing this?

Linear Programming

Input (a "linear program"):

- n variables $x_1, \ldots, x_n$ (take values in $\mathbb{R}$)
- m *non-strict linear inequalities* in these variables (constraints)
  - E.g.: $3x_1 + 4x_2 \leq 6$, $0 \leq x_1 \leq 3$ $x_2 - 3x_3 + 2x_7 = 17$
  - Not allowed (examples): $x_2 x_3 \geq 5$, $x_4 < 2$, $x_5 + \log x_2 \geq 4$
- Possibly a *linear* objective function
  - $\max 2x_3 - 4x_5$, $\min \frac{5}{2}x_4 + x_2$, $\ldots$

Goals:

- Feasibility: Find values for x's that satisfy all constraints
- Optimization: Find feasible solutions maximizing/minimizing objective function

Both achievable in polynomial time, reasonably fast!

Planning your week as an LP

Variables: $\boldsymbol{P}, \boldsymbol{E}, \boldsymbol{S}$

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$$\textbf{max} \quad 2\boldsymbol{P} + \boldsymbol{E}$$

Planning your week as an LP

Variables: P, E, S

$$\begin{aligned} \max \quad & 2P + E \\ \text{subject to} \quad & E \geq 56 \\ & S \geq 60 \\ & 2S + E - 3P \geq 150 \\ & P + E \geq 70 \end{aligned}$$

Planning your week as an LP

Variables: P, E, S

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& P + E \geq 70 \\
& P + S + E = 168 \\
& P \geq 0 \\
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\end{aligned}
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$$

When using an LP to model your problem, need to be sure that *all* aspects of your problem included!

Operations Research-style Example

Four different manufacturing plants for making cars:

	labor	materials	pollution
Plant 1	2	3	15
Plant 2	3	4	10
Plant 3	4	5	9
Plant 4	5	6	7

Operations Research-style Example

Four different manufacturing plants for making cars:

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- Need to produce at least **400** cars at plant 3 (labor agreement)
- Have **3300** total hours of labor, **4000** units of material
- Environmental law: produce at most **12000** pollution
- Make as many cars as possible

OR example as an LP

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Objective: **max** $x_1 + x_2 + x_3 + x_4$

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$$x_3 \geq 400$$

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$$2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 3300$$

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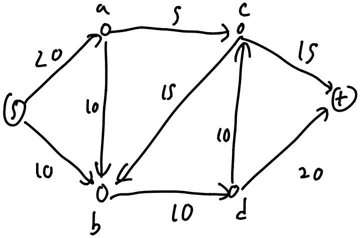
Max Flow as LP



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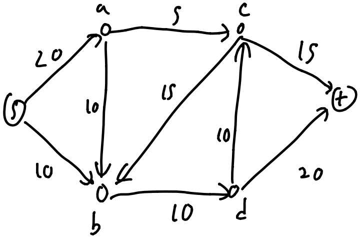

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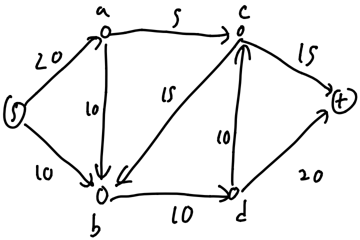

Variables: $f(e)$ for all $e \in E$

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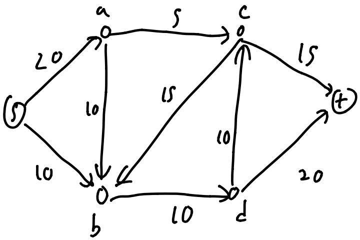

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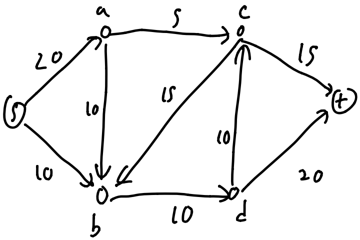

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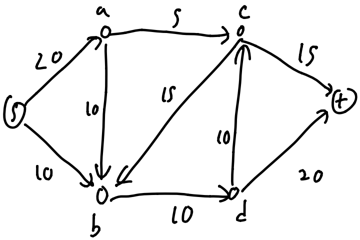

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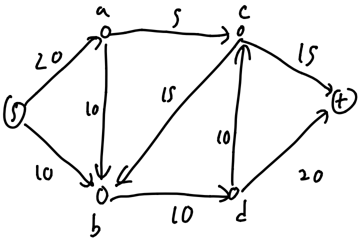

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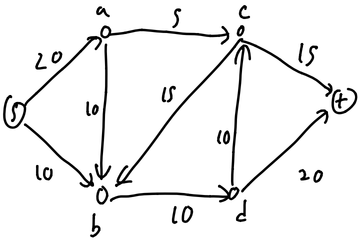

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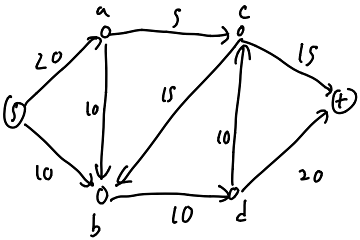

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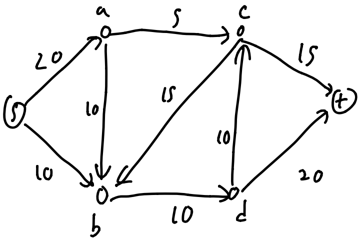

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So can solve max-flow and min-cut (slower) by using generic LP solver

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Generalization of max-flow with multiple commodities that can't mix, but use up same capacity

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Setup:

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- k source-sink pairs $\{(s_i, t_i)\}_{i \in [k]}$

Goal: send flow of commodity i from s_i to t_i, max total flow sent across all commodities

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Concurrent Flow

Multicommodity flow, but:

- Also given *demands* $d : [k] \to \mathbb{R}_{\geq 0}$
- Question: Is there a multicommodity flow that sends at least $d(i)$ commodity-i flow from s_i to t_i for all $i \in [k]$?

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$$f_i(e) \geq 0 \qquad \forall e \in E,\ \forall i \in [k]$$

$$\sum_v f_i(s_i,v) - \sum_v f_i(v,s_i) \geq d(i) \qquad \forall i \in [k]$$

Maximum Concurrent Flow

If answer is no: how much do we need to scale down demands so that there is a multicommodity flow?

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Variables:

- $f_i(e)$ for all $e \in E$ and for all $i \in [k]$.
- λ

Objective: $\max \lambda$

Constraints:

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$$\sum_{i=1}^{k} f_i(e) \le c(e) \qquad \forall e \in E$$

$$f_i(e) \ge 0 \qquad \forall e \in E,\ \forall i \in [k]$$

$$\sum_v f_i(s_i,v) - \sum_v f_i(v,s_i) \ge \lambda d(i) \qquad \forall i \in [k]$$

Shortest $s - t$ path

Very surprising LP!
Variables: d_v for all $v \in V$: shortest-path distance from s to v

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Base case: $i = 0$ ✓
Inductive step: $d_{v_i}^* \leq d_{v_{i-1}}^* + \ell(v_{i-1}, v_i) \leq d(s, v_{i-1}) + \ell(v_{i-1}, v_i) = d(s, v_i)$

Algorithms for LPs

Geometry

To get intuition: think of LPs *geometrically*

- Space: $\mathbb{R}^{\boldsymbol{n}}$ (one dimension per variable
- Linear constraint: halfspace (one side of a hyperplane)
- Feasible region: intersection of halfspaces. *Convex Polytope* (usually just called a *polytope*)

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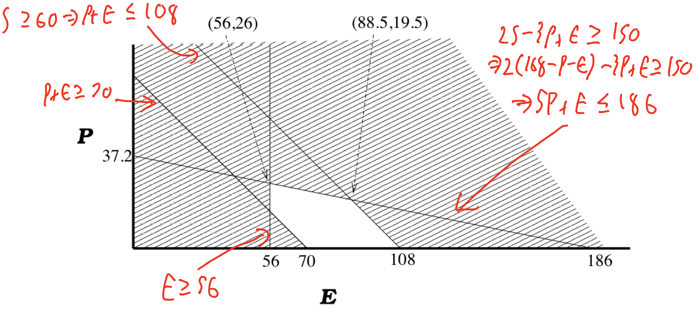
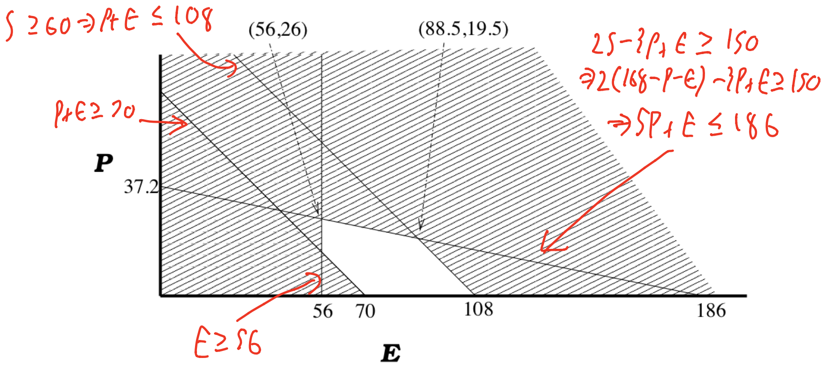
Example: planning your week

- **3** variables S, P, E so $\mathbb{R}^3$
- But $S + P + E = 168 \implies S = 168 - P - E$
- Make this substitution, get $\mathbb{R}^2$



Geometry (cont'd)



Objective: feasible solution "furthest" along specified direction

- **max P: (56, 26)**
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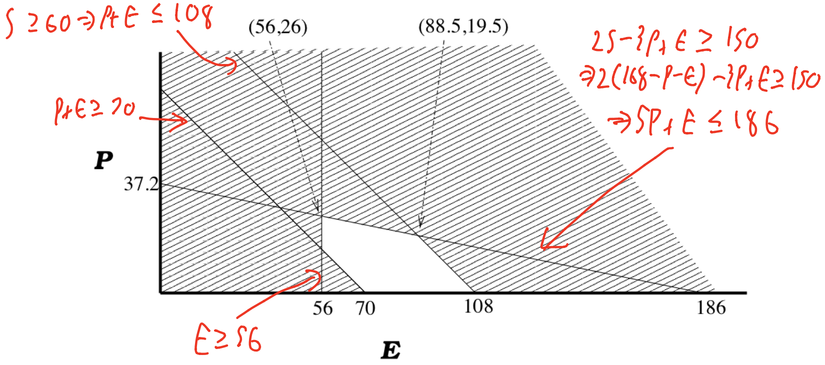

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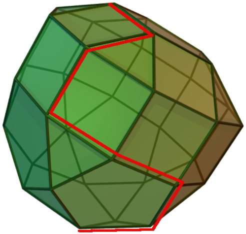
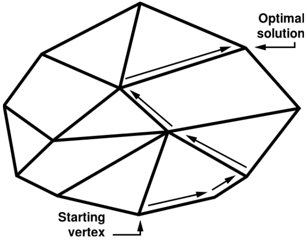
Main theorem: optimal solution is always at a "corner" (also called a "vertex")

Simplex Algorithm [Dantzig 1940's]

Initialize $\vec{x}$ to an arbitrary corner
while(a neighboring corner $\vec{x}'$ of $\vec{x}$ has better objective value) {
    $\vec{x} \leftarrow \vec{x}'$
}
return $\vec{x}$





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  - Much of AMS LP course really about simplex: traditionally favorite algorithm of people who want to actually solve LPs
- Some theory to explain discrepancy ("smoothed analysis")

Ellipsoid Algorithm [Khachiyan 1980]

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First polytime algorithm!

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- Start with ellipsoid E containing feasible region P (if it exists)
- Let x be center of E
- While(x not feasible)
  - Find a hyperplane H through x such that all of P on one side
  - Let E' be the half-ellipsoid of E defined by H
  - Find a new ellipsoid $\hat{E}$ containing E' so that $vol(\hat{E}) \leq \left(1 - \frac{1}{n}\right) vol(E)$
  - Let $E = \hat{E}$ and let x be center of $\hat{E}$

Analysis

Extremely complicated!

Geometry of ellipsoids: can always find an ellipsoid containing a half-ellipsoid with at most $(1 - 1/n)$ of the volume of the original

- Using inequality from last time: after n iterations, volume drops by $\left(1 - \frac{1}{n}\right)^n \leq 1/e$ factor
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In practice: horrible.

Interior Point Methods (Karmarkar's Algorithm)

Fast in both theory and practice!