

Lecture 23: NP-Completeness II

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601.433/633 Introduction to Algorithms

Introduction

Last time: Definition of P , NP , reductions, NP -completeness. Proof that Circuit-SAT is NP -complete.

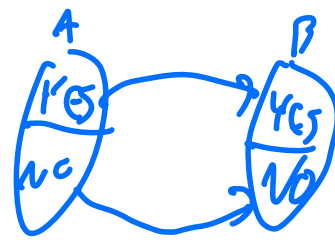
Today: more NP-complete problems.

Definition

A decision problem Q is in NP (*nondeterministic polynomial time*) if there exists a polynomial time algorithm $V(I, X)$ (called the *verifier*) such that

1. If I is a YES-instance of Q , then there is some X (usually called the *witness*, *proof*, or *solution*) with size polynomial in $|I|$ so that $V(I, X) = \text{YES}$.
2. If I is a NO-instance of Q , then $V(I, X) = \text{NO}$ for all X .

Reductions



Definition

A *Many-one* or *Karp* reduction from **A** to **B** is a function f which takes arbitrary instances of **A** and transforms them into instances of **B** so that

1. If x is a YES-instance of **A** then $f(x)$ is a YES-instance of **B**.
2. If x is a NO-instance of **A** then $f(x)$ is a NO-instance of **B**.
3. f can be computed in polynomial time.

Definition

Problem **Q** is *NP-hard* if $Q' \leq_p Q$ for all problems Q' in *NP*. Problem **Q** is *NP-complete* if it is *NP-hard* and in *NP*.

Circuit-SAT

Definition

Circuit-SAT: Given a boolean circuit of AND, OR, and NOT gates, with a single output and no loops (some inputs might be hardwired), is there a way of setting the inputs so that the output of the circuit is **1**?

Theorem

Circuit-SAT is **NP**-complete.

3-SAT

Boolean formula:

- ▶ Boolean variables x_1, \dots, x_n
- ▶ Literal: variable x_i or negation \bar{x}_i
- ▶ AND: \wedge OR: \vee
- ▶ $x_1 \vee (\bar{x}_5 \wedge x_7) \wedge (\bar{x}_2 \vee (x_6 \wedge \bar{x}_3)) \dots$

Conjunctive normal form (CNF): AND of ORs (clauses)

- ▶ $(x_1 \vee \bar{x}_2 \vee \bar{x}_4) \wedge (x_2 \vee x_3) \wedge (x_1 \vee x_4 \vee \bar{x}_6) \dots$

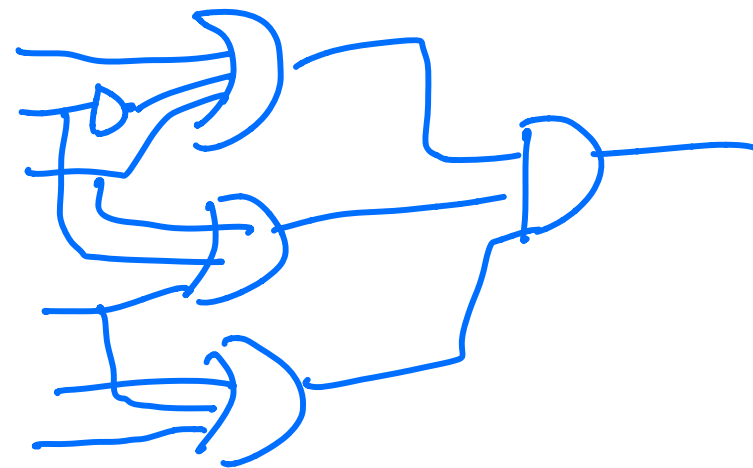
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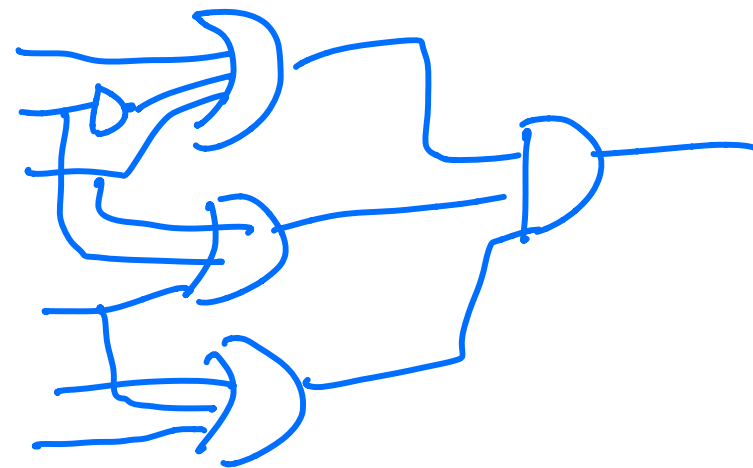
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Definition

3-SAT: Instance is 3CNF formula ϕ (every clause has ≤ 3 literals). YES if there is assignment where ϕ evaluates to True (satisfying assignment), NO otherwise.

3-SAT

Theorem

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- ▶ Don't need to show that $\mathbf{A} \leq_p \text{3-SAT}$ for arbitrary $\mathbf{A} \in \mathbf{NP}$: already know that $\mathbf{A} \leq_p \text{Circuit-SAT}$!

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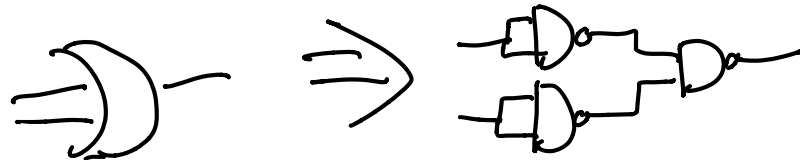
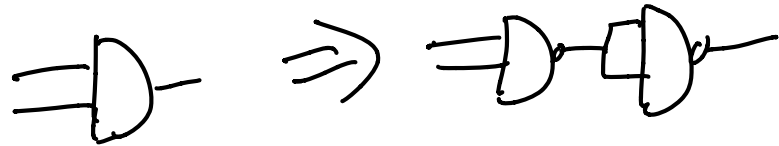
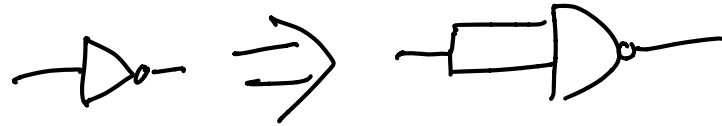
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So start with circuit. Want to transform to 3-CNF formula.

Transformation to NANDs

For simplicity, transform into a circuit with one type of gate: NAND (NOT AND)

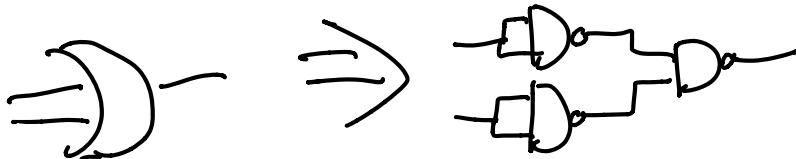
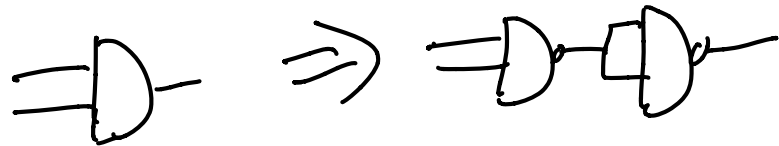
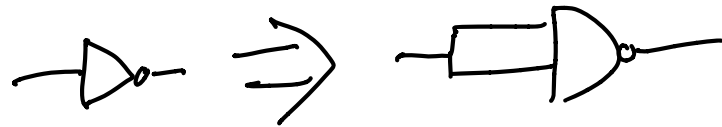
- ▶ AND/OR/NOT universal, but so is just NAND!



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- ▶ AND/OR/NOT universal, but so is just NAND!



So given circuit C , first transform it into NAND-only circuit.

Input:

- ▶ n “input wires” x_1, x_2, \dots, x_n
- ▶ m NAND gates: g_1, \dots, g_m
 - ▶ $g_1 = \mathbf{NAND}(x_1, x_2),$
 $g_2 = \mathbf{NAND}(g_1, x_3), \dots$
- ▶ WLOG, g_m is the “output gate”

Reduction to 3-SAT

So given as input a circuit \mathbf{C} :

- ▶ n “input wires” x_1, x_2, \dots, x_n
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Need to construct many-one reduction f to 3-SAT: in polynomial time, construct 3-CNF formula $f(\mathbf{C})$ such that $f(\mathbf{C})$ has a satisfying assignment if and only if \mathbf{C} has an input where it outputs $\mathbf{1}$.

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Variables: $y_1, y_2, \dots, y_n, y_{n+1}, y_{n+2}, \dots, y_{n+m}$ (one for each wire)

Reduction to 3-SAT

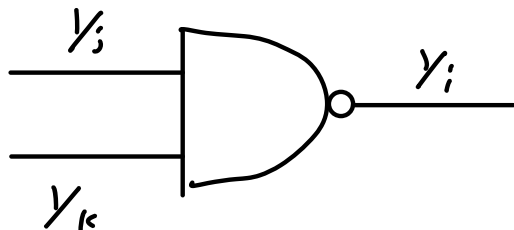
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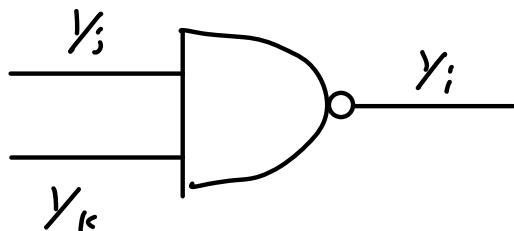
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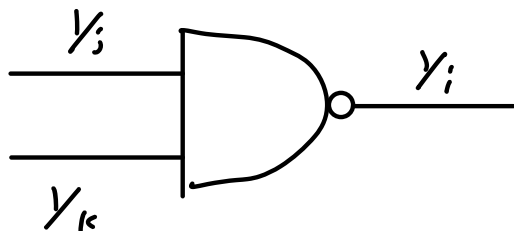
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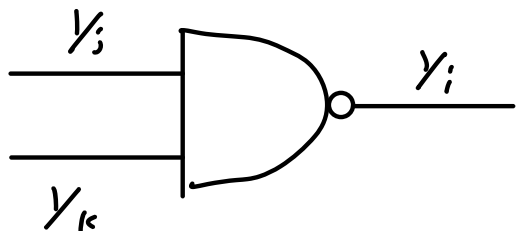
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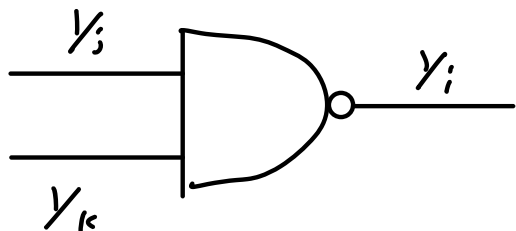
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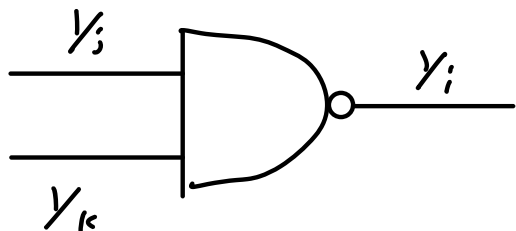
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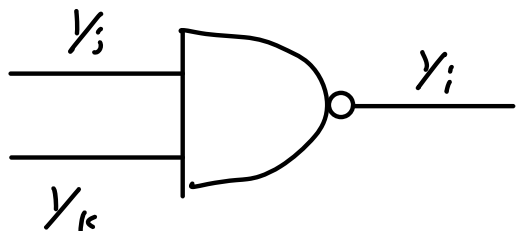
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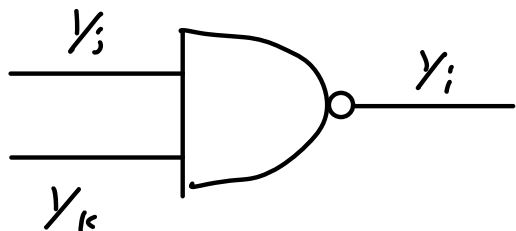
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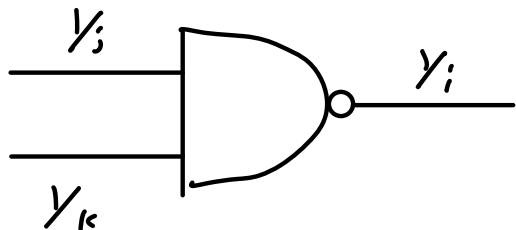
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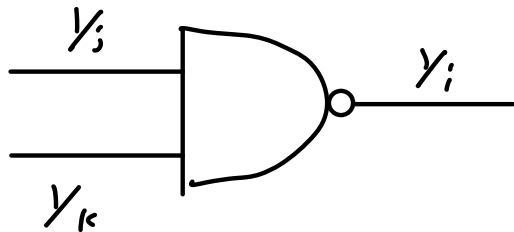
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Also add clause (y_{m+n}) (want output gate to be 1)

Analysis

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Suppose \mathbf{C} YES of Circuit-SAT

- ⇒ \exists setting \mathbf{x} of input wires so $\mathbf{g}_m = \mathbf{1}$
- ⇒ \exists assignment of $\mathbf{y}_1, \dots, \mathbf{y}_{m+n}$ so that all clauses are satisfied:
 - ▶ $\mathbf{y}_i = \mathbf{x}_i$ if $i \leq n$
 - ▶ $\mathbf{y}_i = \mathbf{g}_{i-n}$ if $i > n$
- ⇒ $f(\mathbf{C})$ YES of 3-SAT

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- ⇒ $\mathbf{f}(\mathbf{C})$ YES of 3-SAT

Suppose $\mathbf{f}(\mathbf{C})$ YES of 3-SAT

- ⇒ \exists assignment \mathbf{y} to variables so that all clauses satisfied
- ⇒ \exists setting \mathbf{x} of input wires so $\mathbf{g}_m = \mathbf{1}$:
 - ▶ $\mathbf{x}_i = \mathbf{y}_i$
 - ▶ Output of gate $\mathbf{g}_i = \mathbf{y}_{i+n}$ (by construction)
 - ▶ So $\mathbf{g}_m = \mathbf{1}$ (since (\mathbf{y}_{m+n}) is a clause)
- ⇒ \mathbf{C} a YES instance of Circuit-SAT

General Methodology to Prove Q NP -Complete

1. Show Q is in NP
 - ▶ Can verify witness for YES
 - ▶ Can catch false witness for NO (or contrapositive: if witness is verified, then a YES instance)
2. Find some NP -hard problem A . Reduce *from A to Q* :
 - ▶ Given instance I of A , turn into $f(I)$ of Q (in time polynomial in $|I|$)
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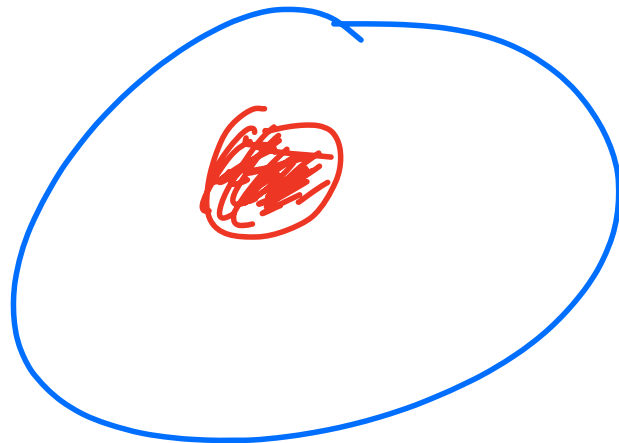
- ▶ Careful about direction of reduction!!!!
- ▶ Need to handle *arbitrary* instances of A , but can turn into very structured instances of Q
- ▶ Often easiest to prove NO direction via contrapositive, to turn into statement about YES:
 - ▶ I YES of $A \implies f(I)$ YES of Q
 - ▶ $f(I)$ YES of $Q \implies I$ YES of A
 - ▶ So proving “both directions”, but reduction only in one direction.

CLIQUE

Definition: A *clique* in an undirected graph $G = (V, E)$ is a set $S \subseteq V$ such that $\{u, v\} \in E$ for all $u, v \in S$

Definition (CLIQUE)

Instance is a graph $G = (V, E)$ and an integer k . YES if G contains a clique of size at least k , NO otherwise.



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Definition (CLIQUE)

Instance is a graph $G = (V, E)$ and an integer k . YES if G contains a clique of size at least k , NO otherwise.

Theorem

CLIQUE is *NP*-complete.

In *NP*:

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Theorem

CLIQUE is *NP*-complete.

In *NP*:

- ▶ Witness: $S \subseteq V$
- ▶ Verifier: Checks if S is a clique and $|S| \geq k$
 - ▶ If (G, k) a YES instance: there is a clique S of size $\geq k$ on which verifier returns YES
 - ▶ If (G, k) a NO instance: S cannot be clique of size $\geq k$, so verifier always returns NO

CLIQUE is *NP*-hard

Prove by reducing 3-SAT to CLIQUE

- ▶ For arbitrary $A \in NP$, would have $A \leq_p \text{Circuit-SAT} \leq_p 3\text{-SAT} \leq_p \text{CLIQUE}$

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Given 3-SAT formula F (with n variables and m clauses), set $k = m$ and create graph $G = (V, E)$:

- ▶ For every clause of F , for every satisfying assignment to the clause, create vertex
- ▶ Add an edge between consistent assignments

CLIQUE is NP -hard

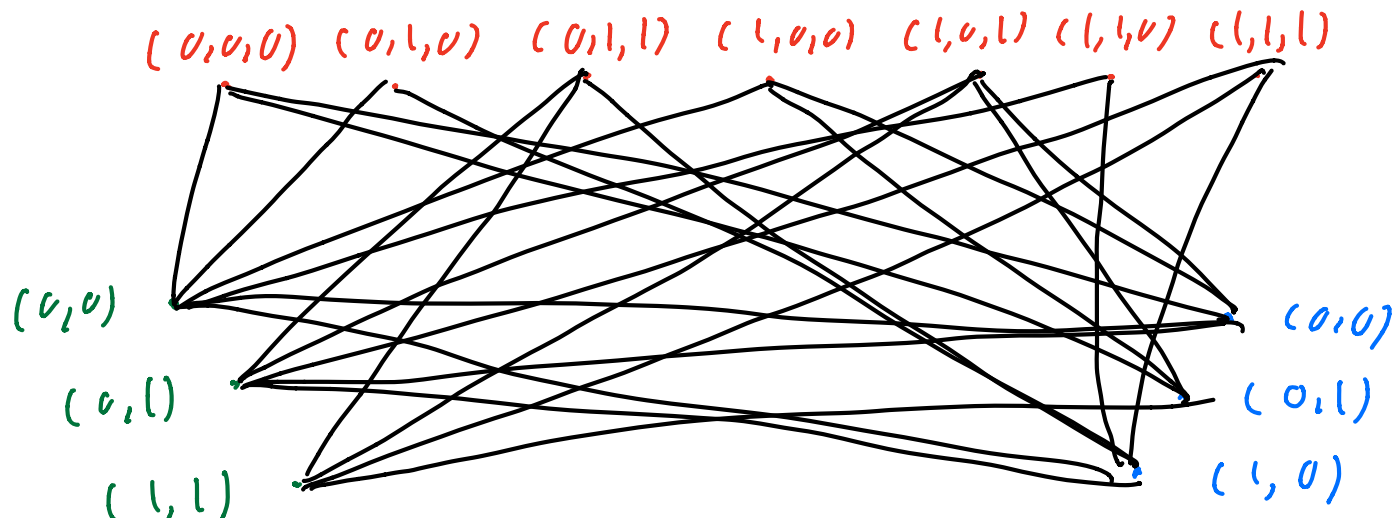
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Example: $F = (x_1 \vee x_2 \vee \bar{x}_4) \wedge (\bar{x}_3 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_3)$



3-SAT to CLIQUE reduction analysis

Polytime: ✓

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If **F** YES of 3-SAT:

- ▶ There is some satisfying assignment \mathbf{x}
- ▶ For every clause, choose vertex corresponding to \mathbf{x} . Let \mathbf{S} be chosen vertices
- ▶ $|\mathbf{S}| = m = k$, and clique since all consistent (since all from \mathbf{x})

⇒ (G, k) YES of CLIQUE

3-SAT to CLIQUE reduction analysis

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⇒ (\mathbf{G}, \mathbf{k}) YES of CLIQUE

If (\mathbf{G}, \mathbf{k}) YES of CLIQUE:

- ▶ There is some clique \mathbf{S} of size $\mathbf{k} = \mathbf{m}$
- ▶ Must contain exactly one vertex from each clause (since clique of size \mathbf{m})
- ▶ Since clique, all assignments consistent ⇒ there is an assignment that satisfies all clauses

⇒ **F** YES of 3-SAT

INDEPENDENT SET

Definition: $S \subseteq V$ is an *independent set* in $G = (V, E)$ if $\{u, v\} \notin E$ for all $u, v \in S$

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Instance is graph $G = (V, E)$ and integer k . YES if G has an independent set of size $\geq k$, NO otherwise.

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- ▶ If (G, k) a YES instance then such an S exists \implies verifier returns YES on it.
- ▶ If (G, k) a NO then verifier will return NO on every S .

INDEPENDENT SET is *NP*-hard

Reduce from:

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- ▶ Given instance (G, k) of CLIQUE, create “complement graph” H : same vertex set, with $\{u, v\} \in E(H)$ if and only if $\{u, v\} \notin E(G)$
- ▶ Instance (H, k) of INDEPENDENT SET

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- ▶ Instance (H, k) of INDEPENDENT SET

If (G, k) YES of CLIQUE:

\implies Clique $S \subseteq V$ of G with $|S| \geq k$

$\implies S$ an independent set in H

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- ▶ Given instance (\mathbf{G}, k) of CLIQUE, create “complement graph” \mathbf{H} : same vertex set, with $\{u, v\} \in \mathbf{E}(\mathbf{H})$ if and only if $\{u, v\} \notin \mathbf{E}(\mathbf{G})$
- ▶ Instance (\mathbf{H}, k) of INDEPENDENT SET

If (\mathbf{G}, k) YES of CLIQUE:

\implies Clique $\mathbf{S} \subseteq \mathbf{V}$ of \mathbf{G} with $|\mathbf{S}| \geq k$

\implies \mathbf{S} an independent set in \mathbf{H}

If (\mathbf{H}, k) YES of INDEPENDENT SET:

\implies Independent set $\mathbf{S} \subseteq \mathbf{V}$ in \mathbf{H} with $|\mathbf{S}| \geq k$

\implies \mathbf{S} a clique in \mathbf{G}

VERTEX COVER

Definition: $S \subseteq V$ is a *vertex cover* of $G = (V, E)$ if $S \cap e \neq \emptyset$ for all $e \in E$

Definition (VERTEX COVER)

Instance is graph $G = (V, E)$, integer k . YES if G has a vertex cover of size $\leq k$, NO otherwise.

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VERTEX COVER is *NP*-hard

Reduce from INDEPENDENT SET

- ▶ Given instance $(G = (V, E), k)$ of INDEPENDENT SET, create instance $(G, n - k)$ of VERTEX COVER (where $n = |V|$)

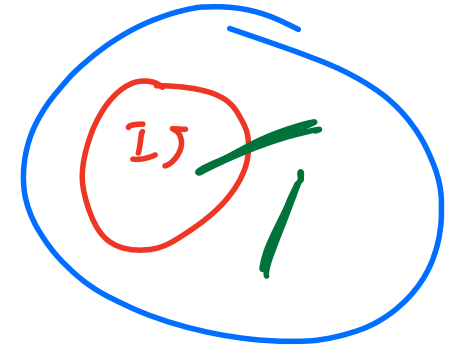
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Reduce from INDEPENDENT SET

- ▶ Given instance $(G = (V, E), k)$ of INDEPENDENT SET, create instance $(G, n - k)$ of VERTEX COVER (where $n = |V|$)

If (G, k) a YES instance of INDEPENDENT SET:

- ⇒ G has an independent set S with $|S| \geq k$
- ⇒ $V \setminus S$ a vertex cover of G of size $\leq n - k$
- ⇒ $(G, n - k)$ a YES instance of VERTEX COVER



VERTEX COVER is *NP*-hard

Reduce from INDEPENDENT SET

- ▶ Given instance $(\mathbf{G} = (\mathbf{V}, \mathbf{E}), k)$ of INDEPENDENT SET, create instance $(\mathbf{G}, n - k)$ of VERTEX COVER (where $n = |\mathbf{V}|$)

If (\mathbf{G}, k) a YES instance of INDEPENDENT SET:

- ⇒ \mathbf{G} has an independent set \mathbf{S} with $|\mathbf{S}| \geq k$
- ⇒ $\mathbf{V} \setminus \mathbf{S}$ a vertex cover of \mathbf{G} of size $\leq n - k$
- ⇒ $(\mathbf{G}, n - k)$ a YES instance of VERTEX COVER

If $(\mathbf{G}, n - k)$ a YES instance of VERTEX COVER:

- ⇒ \mathbf{G} has a vertex cover \mathbf{S} of size at most $n - k$
- ⇒ $\mathbf{V} \setminus \mathbf{S}$ an independent set of \mathbf{G} of size at least k
- ⇒ (\mathbf{G}, k) a YES instance of INDEPENDENT SET