Lecture 24: Approximation Algorithms

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601.433/633 Introduction to Algorithms

Introduction

What should we do if a problem is NP-hard?

- Give up on efficiency?
- Give up on correctness?
- Give up on worst-case analysis?

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Popular answer: *approximation algorithms* (one of my main research areas!)

- Give up on correctness, but in a provable, bounded way.
- Applies to optimization problems only (not pure decision problems)
- Has to run in polynomial time, but can return answer that is *approximately* correct.

Main Definition

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Let $\mathcal{A}$ be some (minimization) problem, and let I be an instance of that problem. Let $OPT(I)$ be the cost of the optimal solution on that instance. Let ALG be a polynomial-time algorithm for $\mathcal{A}$, and let $ALG(I)$ denote the cost of the solution returned by ALG on instance I. Then we say that ALG is an α-*approximation* if

$$\frac{ALG(I)}{OPT(I)} \leq \alpha$$

for all instances I of $\mathcal{A}$.

- Approximation always at least 1
- For maximization, can either require $\frac{ALG(I)}{OPT(I)} \geq \alpha$ (where $\alpha < 1$) or $\frac{OPT(I)}{ALG(I)} \leq \alpha$ (where $\alpha > 1$)

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- For maximization, can either require $\frac{ALG(I)}{OPT(I)} \geq \alpha$ (where $\alpha < 1$) or $\frac{OPT(I)}{ALG(I)} \leq \alpha$ (where $\alpha > 1$)
- Also gives "fine-grained" complexity: not all NP-hard problems are equally hard!

Vertex Cover

Definition: $S \subseteq V$ is a *vertex cover* of $G = (V, E)$ if $S \cap e \neq \emptyset$ for all $e \in E$

Definition (VERTEX COVER)

Instance is graph $G = (V, E)$. Find vertex cover S, minimize $|S|$.

Last time: VERTEX COVER $\mathbf{NP}$-hard (reduction from INDEPENDENT SET)

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Last time: Vertex Cover ***NP***-hard (reduction from Independent Set)

So cannot expect to compute a minimum vertex cover efficiently. What about an *approximately* minimum vertex cover?

- Not an approximate vertex cover: still needs to be an actual vertex cover!

Obvious Algorithm 1

$\boldsymbol{S} = \varnothing$
while there is at least one uncovered edge {
    Pick arbitrary vertex $\boldsymbol{v}$ incident on at least one uncovered edge
    Add $\boldsymbol{v}$ to $\boldsymbol{S}$
}

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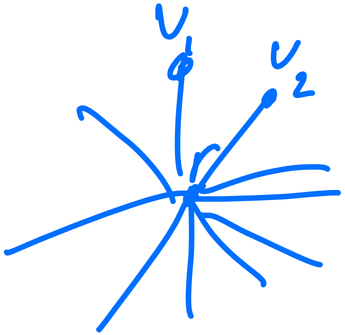
Not a good approximation: star graph.

- $OPT = 1$
- $ALG = n - 1$



Obvious Algorithm 2

$S = \emptyset$
while there is at least one uncovered edge {
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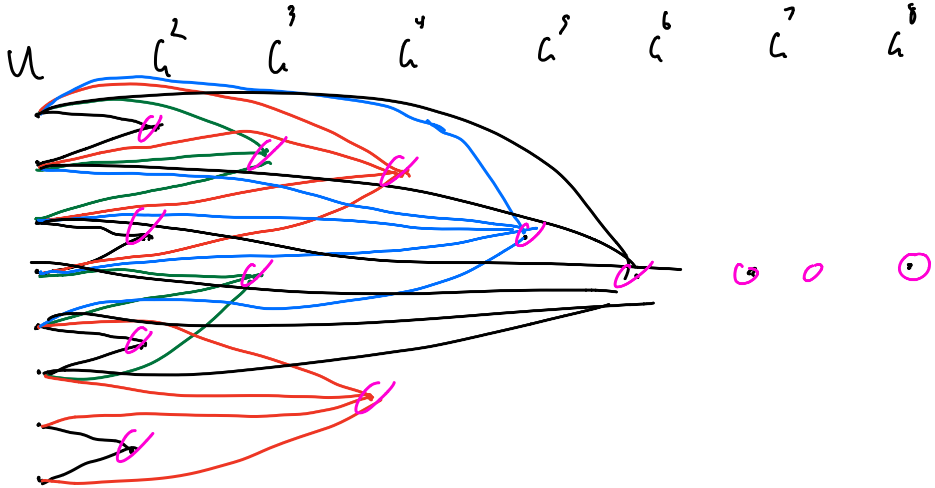
- $|U| = t$
- For all $i \in \{2, 3, \ldots, t\}$, divide U into $\lfloor t/i \rfloor$ disjoint sets of size i: $G^i_1, G^i_2, \ldots, G^i_{\lfloor t/i \rfloor}$
- Add vertex for each set, edge to all elements.



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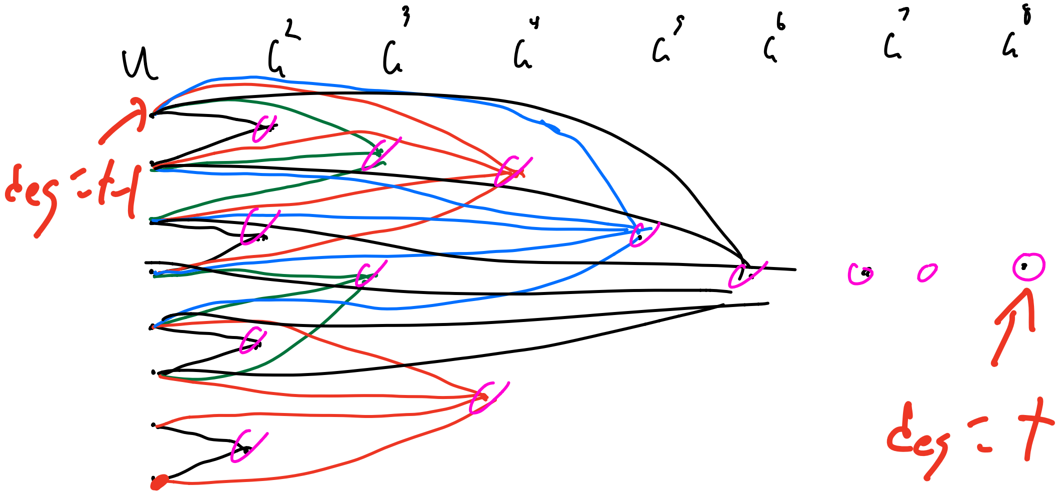

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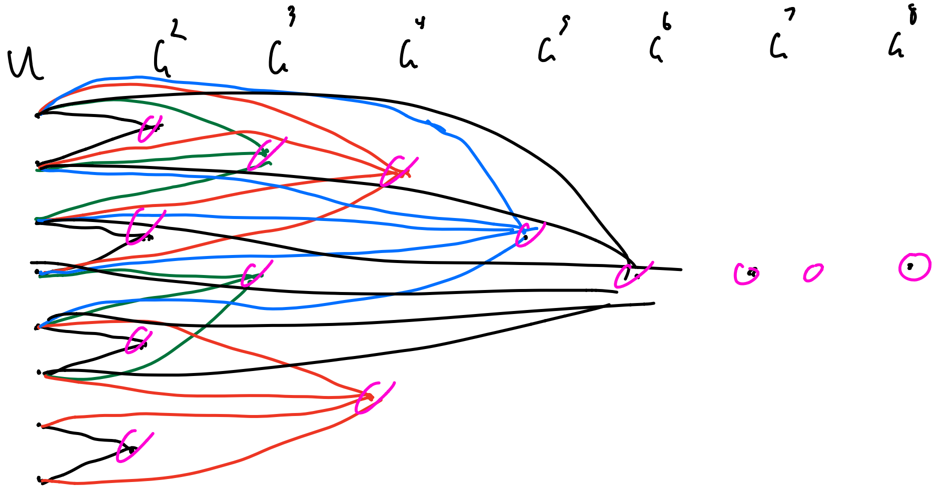

$OPT = t$

$ALG = \sum_{i=2}^{t} \lfloor \frac{t}{i} \rfloor \geq \sum_{i=2}^{t} \left(\frac{1}{2} \cdot \frac{t}{i} \right) = \frac{t}{2} \sum_{i=2}^{t} \frac{1}{i} = \Omega(t \log t)$

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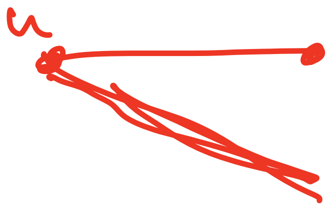

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Suppose algorithm take $\boldsymbol{k}$ iterations. Let $\boldsymbol{L}$ be *edges* chosen by the algorithm, so $|\boldsymbol{L}| = \boldsymbol{k}$.
$\implies |\boldsymbol{S}| = 2\boldsymbol{k}$

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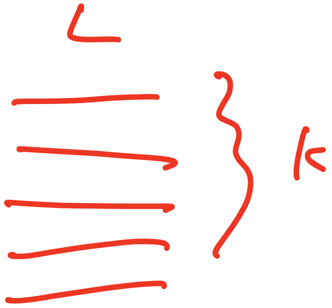
L has structure: it is a matching!



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$\implies ALG/OPT \leq 2$.

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Question: Is this enough?

- Let $OPT(LP)$ denote value of optimal LP solution: does $OPT(LP) = OPT$?

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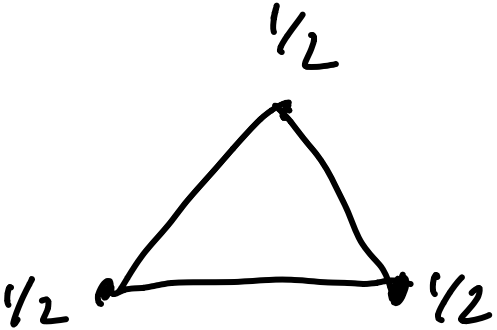

- $OPT = 2$
- $OPT(LP) = 3/2$

LP Structure

$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & x_u + x_v \geq 1 && \forall \{u, v\} \in E \\ & 0 \leq x_u \leq 1 && \forall u \in V \end{aligned}$$

Lemma

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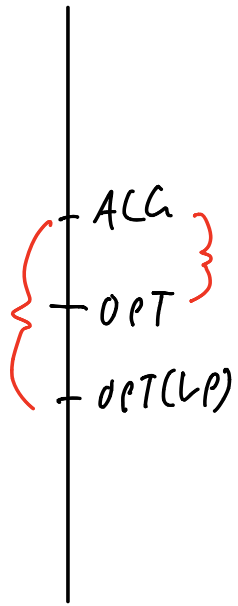

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- Solve LP to get x^* (so $\sum_{v \in V} x_v^* = OPT(LP)$)
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$|S| \leq 2 \cdot OPT$.

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$$|S| = \sum_{v\in S} 1 \leq \sum_{v\in S} 2x_v^* \leq 2\sum_{v\in V} x_v^*$$
$$= 2\cdot OPT(LP) \leq 2\cdot OPT$$ □

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Higher level: LP provides *lower bound* on OPT. Often main difficulty!

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Proved VERTEX COVER ***NP***-hard by reduction from INDEPENDENT SET:

- Polytime algorithm for VERTEX COVER $\implies$ polytime algorithm for INDEPENDENT SET

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There is a notion of "approximation-preserving reduction", but it is more involved than a normal reduction.

Max-E3SAT

Recall 3-SAT: CNF formula (AND of ORs) where every clause has $\leq \mathbf{3}$ literals

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Easy *randomized* algorithm: Choose random assignment!

- For each variable x_i, set $x_i = T$ with probability $1/2$ and F with probability $1/2$

Max-E3SAT: Analysis

Algorithm: Choose random assignment

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Theorem (Håstad '01)

Assuming $P \neq NP$, for all constant $\epsilon > 0$ there is no polytime $\left(\frac{7}{8} + \epsilon\right)$-approximation for Max-E3SAT.