Lecture 25: Online Algorithms

Michael Dinitz

December 3, 2024
601.433/633 Introduction to Algorithms

Introduction

Class until now: difficulty was *computational power*

Today: difficulty is *lack of information*

Online:

- Input / data arrives *over time*
- Need to make decisions without knowing future

Ski Rental Problem

Want to go skiing, but don't know how many times you'll be able to go this year.

Should you rent or buy?

- Renting skis: \$50
- Buying skis: \$500
- Every day you ski and haven't yet bought, need to decide: rent or buy?

Ski Rental Problem

Want to go skiing, but don't know how many times you'll be able to go this year.

Should you rent or buy?

- Renting skis: \$50
- Buying skis: \$500
- Every day you ski and haven't yet bought, need to decide: rent or buy?

Buy right away:

Ski Rental Problem

Want to go skiing, but don't know how many times you'll be able to go this year.

Should you rent or buy?

- Renting skis: \$50
- Buying skis: \$500
- Every day you ski and haven't yet bought, need to decide: rent or buy?

Buy right away:

- If you only ski once, should have rented (\$50), instead bought (\$500)

Ski Rental Problem

Want to go skiing, but don't know how many times you'll be able to go this year.

Should you rent or buy?

- Renting skis: \$50
- Buying skis: \$500
- Every day you ski and haven't yet bought, need to decide: rent or buy?

Buy right away:

- If you only ski once, should have rented (\$50), instead bought (\$500)

Never buy:

Ski Rental Problem

Want to go skiing, but don't know how many times you'll be able to go this year.

Should you rent or buy?

- Renting skis: \$50
- Buying skis: \$500
- Every day you ski and haven't yet bought, need to decide: rent or buy?

Buy right away:

- If you only ski once, should have rented (\$50), instead bought (\$500)

Never buy:

- What if you ski $M \approx \infty$ times?
- Should have bought (\$500), instead rented ($M \cdot \$50$)

Ski Rental Problem

Want to go skiing, but don't know how many times you'll be able to go this year.

Should you rent or buy?

- Renting skis: \$50
- Buying skis: \$500
- Every day you ski and haven't yet bought, need to decide: rent or buy?

Buy right away:

- If you only ski once, should have rented (\$50), instead bought (\$500)

Never buy:

- What if you ski $M \approx \infty$ times?
- Should have bought (\$500), instead rented ($M \cdot \$50$)

What's the right strategy (for these costs)?

Better Late Than Never

Rent until you realize you should have bought!

BLTN: Rent **9** times, buy on **10**'th.

Better Late Than Never

Rent until you realize you should have bought!

BLTN: Rent **9** times, buy on **10**'th.

If ski $\leq$ **9** times:

Better Late Than Never

Rent until you realize you should have bought!

BLTN: Rent **9** times, buy on **10**'th.

If ski $\leq$ **9** times: optimal

Better Late Than Never

Rent until you realize you should have bought!

BLTN: Rent $\mathbf{9}$ times, buy on $\mathbf{10}$'th.

If ski $\leq \mathbf{9}$ times: optimal

If ski $\geq \mathbf{10}$ times:

Better Late Than Never

Rent until you realize you should have bought!

BLTN: Rent $\mathbf{9}$ times, buy on $\mathbf{10}$'th.

If ski $\leq \mathbf{9}$ times: optimal

If ski $\geq \mathbf{10}$ times:

- $\boldsymbol{ALG} = \mathbf{450} + \mathbf{500} = \mathbf{950}$
- $\boldsymbol{OPT} = \mathbf{500}$

Better Late Than Never

Rent until you realize you should have bought!

BLTN: Rent $\mathbf{9}$ times, buy on $\mathbf{10}$'th.

If ski $\leq \mathbf{9}$ times: optimal

If ski $\geq \mathbf{10}$ times:

- $\boldsymbol{ALG} = \mathbf{450} + \mathbf{500} = \mathbf{950}$
- $\boldsymbol{OPT} = \mathbf{500}$

Never more than twice (actually $\frac{\mathbf{19}}{\mathbf{10}}$ times) what we should have done!

Competitive Ratio

Definition

The *competitive ratio* of algorithm $\boldsymbol{ALG}$ is the maximum over all inputs/futures $\boldsymbol{\sigma}$ of

$$\frac{\boldsymbol{ALG}(\boldsymbol{\sigma})}{\boldsymbol{OPT}(\boldsymbol{\sigma})},$$

where $\boldsymbol{ALG}(\boldsymbol{\sigma})$ is the cost of $\boldsymbol{ALG}$ on $\boldsymbol{\sigma}$ and $\boldsymbol{OPT}(\boldsymbol{\sigma})$ is the optimal cost for $\boldsymbol{\sigma}$ (knowing the future).

Competitive Ratio

Definition

The *competitive ratio* of algorithm $\boldsymbol{ALG}$ is the maximum over all inputs/futures σ of

$$\frac{\boldsymbol{ALG}(\sigma)}{\boldsymbol{OPT}(\sigma)},$$

where $\boldsymbol{ALG}(\sigma)$ is the cost of $\boldsymbol{ALG}$ on σ and $\boldsymbol{OPT}(\sigma)$ is the optimal cost for σ (knowing the future).

So on ski rental problem with previous values, competitive ratio is $\frac{19}{10}$.

Ski Rental: Generalized

$\$r$ to rent, $\$b$ to buy. Assume r divides b for simplicity.

Ski Rental: Generalized

$\$r$ to rent, $\$b$ to buy. Assume r divides b for simplicity.

BLTN: Rent $\frac{b}{r} - 1$ times, then buy.

Ski Rental: Generalized

$\$r$ to rent, $\$b$ to buy. Assume r divides b for simplicity.

BLTN: Rent $\frac{b}{r} - 1$ times, then buy.

Theorem

BLTN has competitive ratio at most $2 - \frac{r}{b}$.

Ski Rental: Generalized

$\$r$ to rent, $\$b$ to buy. Assume r divides b for simplicity.

BLTN: Rent $\frac{b}{r} - 1$ times, then buy.

Theorem

BLTN has competitive ratio at most $2 - \frac{r}{b}$.

Case 1: Ski $z \leq \frac{b}{r} - 1$ times

Ski Rental: Generalized

$\$r$ to rent, $\$b$ to buy. Assume r divides b for simplicity.

BLTN: Rent $\frac{b}{r} - 1$ times, then buy.

Theorem

BLTN has competitive ratio at most $2 - \frac{r}{b}$.

Case 1: Ski $z \leq \frac{b}{r} - 1$ times

- $ALG = z \cdot r$
- $OPT = \min(z \cdot r, b) = z \cdot r$

$\Longrightarrow \frac{ALG}{OPT} = 1$

Ski Rental: Generalized

$\$r$ to rent, $\$b$ to buy. Assume r divides b for simplicity.

BLTN: Rent $\frac{b}{r} - 1$ times, then buy.

Theorem

BLTN has competitive ratio at most $2 - \frac{r}{b}$.

Case 1: Ski $z \leq \frac{b}{r} - 1$ times

- $ALG = z \cdot r$
- $OPT = \min(z \cdot r, b) = z \cdot r$

$\Longrightarrow \frac{ALG}{OPT} = 1$

Case 2: Ski $z \geq \frac{b}{r}$ times

Ski Rental: Generalized

$\$r$ to rent, $\$b$ to buy. Assume r divides b for simplicity.

BLTN: Rent $\frac{b}{r} - 1$ times, then buy.

Theorem

BLTN has competitive ratio at most $2 - \frac{r}{b}$.

Case 1: Ski $z \leq \frac{b}{r} - 1$ times

- $ALG = z \cdot r$
- $OPT = \min(z \cdot r, b) = z \cdot r$

$\Longrightarrow \frac{ALG}{OPT} = 1$

Case 2: Ski $z \geq \frac{b}{r}$ times

- $ALG = r \cdot \left(\frac{b}{r} - 1\right) + b = b - r + b = 2b - r$
- $OPT = \min(r \cdot z, b) = b$

$\Longrightarrow \frac{ALG}{OPT} = \frac{2b-r}{b} = 2 - \frac{r}{b}$

Ski Rental: Generalized

$\$r$ to rent, $\$b$ to buy. Assume r divides b for simplicity.

BLTN: Rent $\frac{b}{r} - 1$ times, then buy.

Theorem

BLTN has competitive ratio at most $2 - \frac{r}{b}$.

Case 1: Ski $z \leq \frac{b}{r} - 1$ times

- $ALG = z \cdot r$
- $OPT = \min(z \cdot r, b) = z \cdot r$

$\Longrightarrow \frac{ALG}{OPT} = 1$

Case 2: Ski $z \geq \frac{b}{r}$ times

- $ALG = r \cdot \left(\frac{b}{r} - 1\right) + b = b - r + b = 2b - r$
- $OPT = \min(r \cdot z, b) = b$

$\Longrightarrow \frac{ALG}{OPT} = \frac{2b-r}{b} = 2 - \frac{r}{b}$

So for all inputs / futures, $\frac{ALG}{OPT} \leq 2 - \frac{r}{b}$

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Deterministic ***ALG***: "ski x times, then buy".

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Deterministic ***ALG***: "ski x times, then buy".
Input: ski $x+1$ times.

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Deterministic $\boldsymbol{ALG}$: "ski $\boldsymbol{x}$ times, then buy".
Input: ski $\boldsymbol{x+1}$ times.

Case 1: $\boldsymbol{x \geq b/r}$

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Deterministic $\boldsymbol{ALG}$: "ski $\boldsymbol{x}$ times, then buy".
Input: ski $\boldsymbol{x+1}$ times.

Case 1: $\boldsymbol{x \geq b/r}$

- $\boldsymbol{OPT = \min(b, (x+1)r) = b}$

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Deterministic $\boldsymbol{ALG}$: "ski $\boldsymbol{x}$ times, then buy".
Input: ski $\boldsymbol{x+1}$ times.

Case 1: $\boldsymbol{x \geq b/r}$

- $\boldsymbol{OPT = \min(b, (x+1)r) = b}$
- $\boldsymbol{ALG = xr + b \geq 2b}$

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Deterministic $\boldsymbol{ALG}$: "ski $\boldsymbol{x}$ times, then buy".
Input: ski $\boldsymbol{x+1}$ times.

Case 1: $\boldsymbol{x \geq b/r}$

- $\boldsymbol{OPT = \min(b, (x+1)r) = b}$
- $\boldsymbol{ALG = xr + b \geq 2b}$

$\Longrightarrow$ $\boldsymbol{\frac{ALG}{OPT} \geq 2 > 2 - \frac{r}{b}}$

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Deterministic $\boldsymbol{ALG}$: "ski $\boldsymbol{x}$ times, then buy".
Input: ski $\boldsymbol{x+1}$ times.

Case 1: $\boldsymbol{x \geq b/r}$

- $\boldsymbol{OPT = \min(b, (x+1)r) = b}$
- $\boldsymbol{ALG = xr + b \geq 2b}$

$\Longrightarrow$ $\boldsymbol{\frac{ALG}{OPT} \geq 2 > 2 - \frac{r}{b}}$

Case 2: $\boldsymbol{x \leq \frac{b}{r} - 1}$

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Deterministic $\boldsymbol{ALG}$: "ski $\boldsymbol{x}$ times, then buy".
Input: ski $\boldsymbol{x+1}$ times.

Case 1: $\boldsymbol{x \geq b/r}$

- $\boldsymbol{OPT = \min(b, (x+1)r) = b}$
- $\boldsymbol{ALG = xr + b \geq 2b}$

$\Longrightarrow \quad \boldsymbol{\frac{ALG}{OPT} \geq 2 > 2 - \frac{r}{b}}$

Case 2: $\boldsymbol{x \leq \frac{b}{r} - 1}$

- $\boldsymbol{OPT = \min(b, (x+1)r) = (x+1)r}$

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Deterministic $\boldsymbol{ALG}$: "ski $\boldsymbol{x}$ times, then buy".
Input: ski $\boldsymbol{x+1}$ times.

Case 1: $\boldsymbol{x \geq b/r}$

- $\boldsymbol{OPT = \min(b, (x+1)r) = b}$
- $\boldsymbol{ALG = xr + b \geq 2b}$

$\Longrightarrow \quad \boldsymbol{\frac{ALG}{OPT} \geq 2 > 2 - \frac{r}{b}}$

Case 2: $\boldsymbol{x \leq \frac{b}{r} - 1}$

- $\boldsymbol{OPT = \min(b, (x+1)r) = (x+1)r}$
- $\boldsymbol{ALG = xr + b}$

Lower Bound

Theorem

No (deterministic) algorithm has competitive ratio better than BLTN.

Deterministic $\boldsymbol{ALG}$: "ski $\boldsymbol{x}$ times, then buy".
Input: ski $\boldsymbol{x+1}$ times.

Case 1: $\boldsymbol{x \geq b/r}$

- $\boldsymbol{OPT} = \min(b, (x+1)r) = b$
- $\boldsymbol{ALG} = xr + b \geq 2b$

$\Longrightarrow \frac{ALG}{OPT} \geq 2 > 2 - \frac{r}{b}$

Case 2: $\boldsymbol{x \leq \frac{b}{r} - 1}$

- $\boldsymbol{OPT} = \min(b, (x+1)r) = (x+1)r$
- $\boldsymbol{ALG} = xr + b$

$$\frac{ALG}{OPT} = \frac{xr+b}{(x+1)r} = \frac{xr+b}{xr+r} = 1 + \frac{b-r}{xr+r}$$
$$\geq 1 + \frac{b-r}{(\frac{b}{r}-1)r + r} = 1 + \frac{b-r}{b} = 2 - \frac{r}{b}$$

Elevator Problem

Trying to get up a building: takes $\boldsymbol{E}$ seconds by elevator, $\boldsymbol{S}$ seconds by stairs.

- How long should we wait for the elevator?
- Example: $\boldsymbol{E} = \mathbf{15}, \boldsymbol{S} = \mathbf{45}$.

Elevator Problem

Trying to get up a building: takes $\boldsymbol{E}$ seconds by elevator, $\boldsymbol{S}$ seconds by stairs.

- How long should we wait for the elevator?
- Example: $\boldsymbol{E} = \mathbf{15}, \boldsymbol{S} = \mathbf{45}$.

BLTN: Wait $\boldsymbol{S} - \boldsymbol{E}$ seconds, then give up and take stairs

Elevator Problem

Trying to get up a building: takes $\boldsymbol{E}$ seconds by elevator, $\boldsymbol{S}$ seconds by stairs.

- How long should we wait for the elevator?
- Example: $\boldsymbol{E} = \mathbf{15}, \boldsymbol{S} = \mathbf{45}$.

BLTN: Wait $\boldsymbol{S} - \boldsymbol{E}$ seconds, then give up and take stairs

If elevator arrives at $\boldsymbol{x} \leq \boldsymbol{S} - \boldsymbol{E}$:

Elevator Problem

Trying to get up a building: takes E seconds by elevator, S seconds by stairs.

- How long should we wait for the elevator?
- Example: $E = 15, S = 45$.

BLTN: Wait $S - E$ seconds, then give up and take stairs

If elevator arrives at $x \leq S - E$:

- $OPT = \min(S, x + E) = x + E$
- $ALG = x + E$

$\Longrightarrow \frac{ALG}{OPT} = 1$

Elevator Problem

Trying to get up a building: takes E seconds by elevator, S seconds by stairs.

- How long should we wait for the elevator?
- Example: $E = 15, S = 45$.

BLTN: Wait $S - E$ seconds, then give up and take stairs

If elevator arrives at $x \leq S - E$:

- $OPT = \min(S, x + E) = x + E$
- $ALG = x + E$

$\Longrightarrow \frac{ALG}{OPT} = 1$

If elevator arrives at $x > S - E$:

Elevator Problem

Trying to get up a building: takes E seconds by elevator, S seconds by stairs.

- How long should we wait for the elevator?
- Example: $E = 15, S = 45$.

BLTN: Wait $S - E$ seconds, then give up and take stairs

If elevator arrives at $x \leq S - E$:

- $OPT = \min(S, x + E) = x + E$
- $ALG = x + E$

$\Longrightarrow \frac{ALG}{OPT} = 1$

If elevator arrives at $x > S - E$:

- $OPT = \min(S, x + E) = S$
- $ALG = (S - E) + S = 2S - E$

$\Longrightarrow \frac{ALG}{OPT} = \frac{2S-E}{S} = 2 - \frac{E}{S}$

Extensions: Randomization

What if our algorithm is allowed to be randomized?

- Choose our buying time from some distribution $\boldsymbol{p}$ over days (*purchase distribution*)
- Adversary knows $\boldsymbol{p}$, but not our random draw from $\boldsymbol{p}$ (*oblivious* adversary).
- Normalize so $\boldsymbol{r} = \mathbf{1}$

Extensions: Randomization

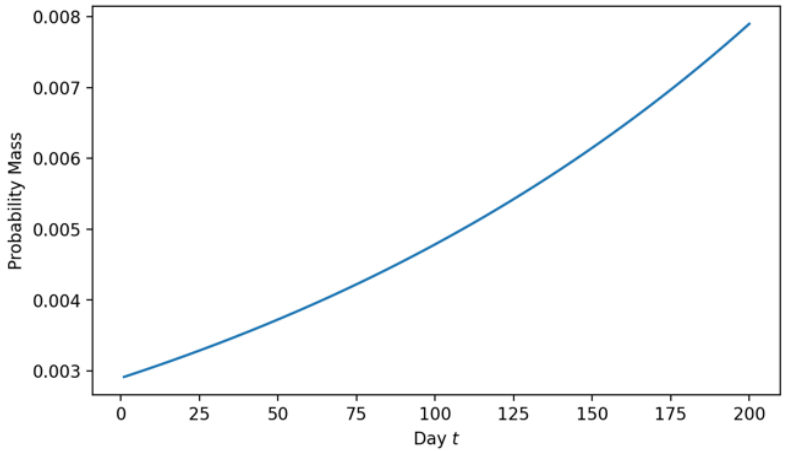
What if our algorithm is allowed to be randomized?

- Choose our buying time from some distribution $\boldsymbol{p}$ over days (*purchase distribution*)
- Adversary knows $\boldsymbol{p}$, but not our random draw from $\boldsymbol{p}$ (*oblivious* adversary).
- Normalize so $\boldsymbol{r} = \mathbf{1}$

Theorem (Karlin, Manasse, McGeoch, Owicki (SODA '90))

If we set $\boldsymbol{p_t} = \left(\frac{b-1}{b}\right)^{b-t} \frac{1}{b\left(1-\left(1-\frac{1}{b}\right)^{b}\right)}$ *for all* $\boldsymbol{t \le b}$, *then*

- $\frac{E[ALG(\sigma)]}{OPT(\sigma)} \le \frac{e}{e-1} \approx \mathbf{1.58}$ *for all* σ, *and*
- *this is optimal.*



Extensions: Randomization

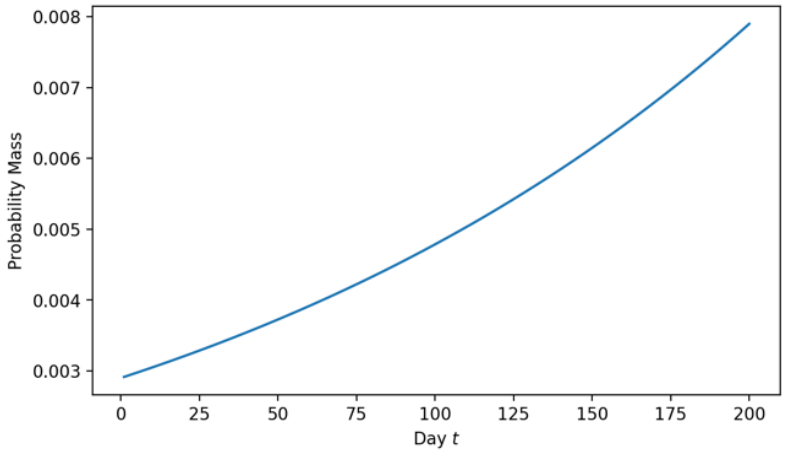
What if our algorithm is allowed to be randomized?

- Choose our buying time from some distribution $\boldsymbol{p}$ over days (*purchase distribution*)
- Adversary knows $\boldsymbol{p}$, but not our random draw from $\boldsymbol{p}$ (*oblivious* adversary).
- Normalize so $\boldsymbol{r} = \mathbf{1}$

Theorem (Karlin, Manasse, McGeoch, Owicki (SODA '90))

If we set $\boldsymbol{p_t} = \left(\frac{b-1}{b}\right)^{b-t} \frac{1}{b\left(1-\left(1-\frac{1}{b}\right)^{b}\right)}$ *for all* $\boldsymbol{t \le b}$, *then*

- $\frac{E[ALG(\sigma)]}{OPT(\sigma)} \le \frac{e}{e-1} \approx \mathbf{1.58}$ *for all* σ, *and*
- *this is optimal.*



Good in expectation, but what about probability of being bad (e.g., worse than BLTN)?

Extensions: Randomization

Pr[worse than BLTN] $\approx$ **0.3775**: quite large!

Extensions: Randomization

$\boldsymbol{Pr}$[worse than BLTN] $\approx \mathbf{0.3775}$: quite large!

Definition

(γ, δ)-tail bound: probability at most δ of having competitive ratio larger than γ

Extensions: Randomization

$\boldsymbol{Pr}$[worse than BLTN] $\approx \mathbf{0.3775}$: quite large!

Definition

(γ, δ)-tail bound: probability at most δ of having competitive ratio larger than γ

We introduced and studied recently: Dinitz, Im, Lavastida, Moseley, Vassilvitskii [SODA '24]

Extensions: Randomization

$\boldsymbol{Pr}$[worse than BLTN] $\approx \mathbf{0.3775}$: quite large!

Definition

(γ, δ)-tail bound: probability at most δ of having competitive ratio larger than γ

We introduced and studied recently: Dinitz, Im, Lavastida, Moseley, Vassilvitskii [SODA '24]

- Given collection of tail bounds, algorithm to compute optimal randomized algorithm satisfying them.
- They can look crazy!

Extensions: Randomization

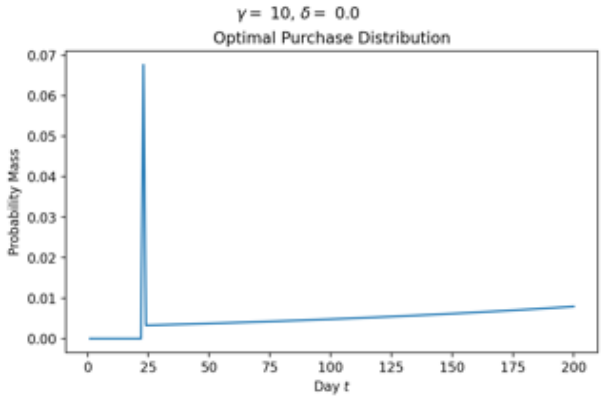
Pr[worse than BLTN] $\approx$ **0.3775**: quite large!

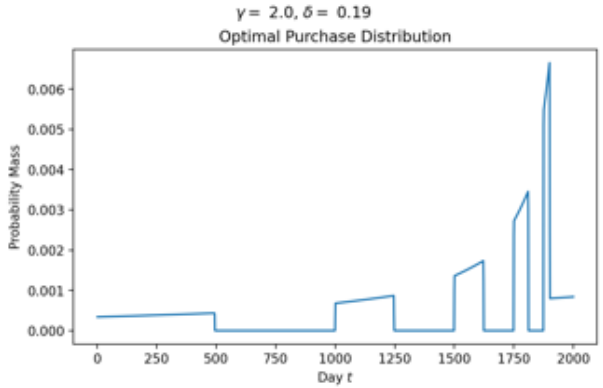
Definition

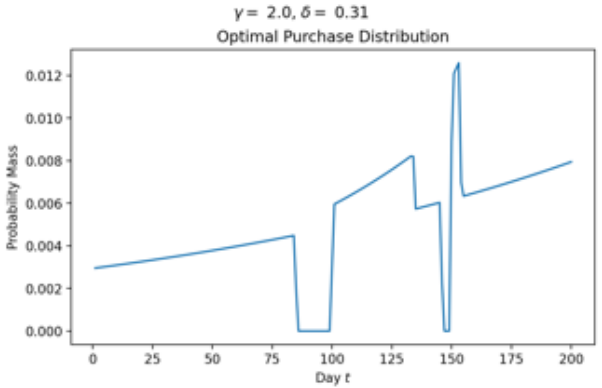
(γ, δ)-tail bound: probability at most δ of having competitive ratio larger than γ

We introduced and studied recently: Dinitz, Im, Lavastida, Moseley, Vassilvitskii [SODA '24]

- Given collection of tail bounds, algorithm to compute optimal randomized algorithm satisfying them.
- They can look crazy!







Paging

Classical problem in computer systems/theory

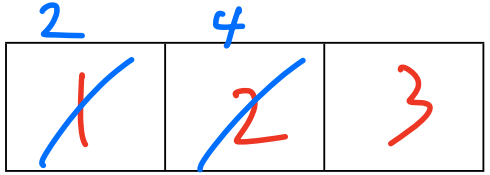
- Disk (slow) with N pages
- Memory (fast) with room for $k < N$ pages
- If OS/application requests a page not in memory: "page fault"
  - Need to bring requested page into memory, *evict* a page from memory (if currently full)
- **Question:** What to evict?

Paging

Classical problem in computer systems/theory

- Disk (slow) with N pages
- Memory (fast) with room for $k < N$ pages
- If OS/application requests a page not in memory: "page fault"
  - Need to bring requested page into memory, *evict* a page from memory (if currently full)
- **Question:** What to evict?

Example: $k = 3$. Requests: $1, 2, 3, 2, 4, 3, 4, 1, 2, 3, 4$



(Convention: initial page faults to fill table don't count: only pay when we *evict* a page)

LRU

Standard algorithm: "Least Recently Used" (LRU)

- Evict page from memory that hasn't been used in the longest time

LRU

Standard algorithm: "Least Recently Used" (LRU)

- Evict page from memory that hasn't been used in the longest time
- Intuition:
  - Want to evict page that's next used furthest in the future. But don't know future!
  - Hope that since it hasn't been used for a long time, won't be requested again for a long time.

LRU

Standard algorithm: "Least Recently Used" (LRU)

- Evict page from memory that hasn't been used in the longest time
- Intuition:
  - Want to evict page that's next used furthest in the future. But don't know future!
  - Hope that since it hasn't been used for a long time, won't be requested again for a long time.

Is this a good algorithm? What's the competitive ratio? Cost = # evictions.

LRU

Standard algorithm: "Least Recently Used" (LRU)

- Evict page from memory that hasn't been used in the longest time
- Intuition:
  - Want to evict page that's next used furthest in the future. But don't know future!
  - Hope that since it hasn't been used for a long time, won't be requested again for a long time.

Is this a good algorithm? What's the competitive ratio? Cost = # evictions.

- $k = 3$, $N = 4$

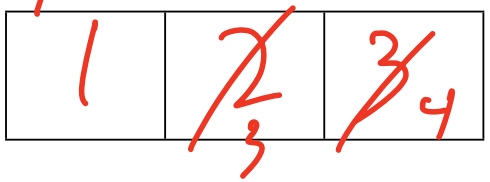
LRU

Standard algorithm: "Least Recently Used" (LRU)

- Evict page from memory that hasn't been used in the longest time
- Intuition:
  - Want to evict page that's next used furthest in the future. But don't know future!
  - Hope that since it hasn't been used for a long time, won't be requested again for a long time.

Is this a good algorithm? What's the competitive ratio? Cost = # evictions.

- $k = 3$, $N = 4$
- Requests: $1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \ldots$



LRU

Standard algorithm: "Least Recently Used" (LRU)

- Evict page from memory that hasn't been used in the longest time
- Intuition:
  - Want to evict page that's next used furthest in the future. But don't know future!
  - Hope that since it hasn't been used for a long time, won't be requested again for a long time.

Is this a good algorithm? What's the competitive ratio? Cost = # evictions.

- $k = 3$, $N = 4$
- Requests: $1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \ldots$

So LRU has competitive ratio $\approx k$

- LRU evicts every time, ***OPT*** evicts 1 out of every k times.

Lower Bound

Theorem

No deterministic algorithm has competitive ratio less than k.

Let ALG be some deterministic algorithm. Set $N = k + 1$

Lower Bound

Theorem

No deterministic algorithm has competitive ratio less than k.

Let ALG be some deterministic algorithm. Set $N = k + 1$
Request sequence:

Lower Bound

Theorem

No deterministic algorithm has competitive ratio less than k.

Let ALG be some deterministic algorithm. Set $N = k + 1$
Request sequence: Whatever is not in memory for ALG!

Lower Bound

Theorem

No deterministic algorithm has competitive ratio less than k.

Let ALG be some deterministic algorithm. Set $N = k + 1$
Request sequence: Whatever is not in memory for ALG!

$\Longrightarrow$ ALG has an eviction every time (after initialization)

Lower Bound

Theorem

No deterministic algorithm has competitive ratio less than k.

Let ALG be some deterministic algorithm. Set $N = k + 1$
Request sequence: Whatever is not in memory for ALG!

$\Longrightarrow$ ALG has an eviction every time (after initialization)

OPT: evict page whose next request is furthest in the future

- Every page in memory needs to be requested before next eviction. So next eviction is in at least k steps.

Lower Bound

Theorem

No deterministic algorithm has competitive ratio less than k.

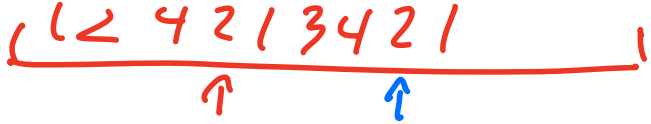
Let ALG be some deterministic algorithm. Set $N = k + 1$
Request sequence: Whatever is not in memory for ALG!

$\Longrightarrow$ ALG has an eviction every time (after initialization)

OPT: evict page whose next request is furthest in the future

- Every page in memory needs to be requested before next eviction. So next eviction is in at least k steps.

$$\Longrightarrow \frac{ALG}{OPT} \geq k$$



Marking Algorithm

Get around lower bound by using *randomization*

- Lower bound argument doesn't apply because can't set request sequence to ask for whatever's not in memory, since that involved randomness! (Oblivious adversary)

Marking Algorithm

Get around lower bound by using *randomization*

- Lower bound argument doesn't apply because can't set request sequence to ask for whatever's not in memory, since that involved randomness! (Oblivious adversary)

Assume memory initially $\mathbf{1, 2, \ldots, k}$.
Set all pages in memory to be "unmarked"

When page requested:

- If already in memory, "mark" it
- If not in memory:
  - If all pages in memory "marked", unmark all
  - Choose an *unmarked* page uniformly at random to evict
  - Bring in new page, mark it

Marking Analysis

Theorem

Expected competitive ratio at most $O(\log k)$*:*

$$\frac{E[ALG(\sigma)]}{OPT(\sigma)} \leq O(\log k) \text{ for all request sequences } \sigma.$$

Marking Analysis

Theorem

Expected competitive ratio at most $O(\log k)$*:*

$$\frac{E[ALG(\sigma)]}{OPT(\sigma)} \leq O(\log k) \text{ for all request sequences } \sigma.$$

Proof sketch for $N = k + 1$: full generality more complicated

Marking Analysis

Theorem

Expected competitive ratio at most $O(\log k)$*:*

$$\frac{E[ALG(\sigma)]}{OPT(\sigma)} \leq O(\log k) \text{ for all request sequences } \sigma.$$

Proof sketch for $N = k + 1$: full generality more complicated

Phase: time between "unmark all" events.

Marking Analysis

Theorem

Expected competitive ratio at most $O(\log k)$*:*

$$\frac{E[ALG(\sigma)]}{OPT(\sigma)} \leq O(\log k) \text{ for all request sequences } \sigma.$$

Proof sketch for $N = k + 1$: full generality more complicated

Phase: time between "unmark all" events.

In each phase:

- $OPT \geq 1$, since all N pages requested

ALG in each phase

Key point: the one page *not* in memory is *uniformly distributed* among all *unmarked* pages.

ALG in each phase

Key point: the one page *not* in memory is *uniformly distributed* among all *unmarked* pages.

When page requested:

- If marked: in memory, no eviction
- If unmarked: if currently i unmarked pages, then $\Pr[\text{eviction}] = \Pr[\text{requested page not in memory}] = 1/i$
  - Becomes marked

ALG in each phase

Key point: the one page *not* in memory is *uniformly distributed* among all *unmarked* pages.

When page requested:

- If marked: in memory, no eviction
- If unmarked: if currently i unmarked pages, then $\Pr[\text{eviction}] = \Pr[\text{requested page not in memory}] = 1/i$
  - Becomes marked

At beginning of phase $i = N$, at end of phase $i = 1$. Goes down by one every time page gets marked.

ALG in each phase

Key point: the one page *not* in memory is *uniformly distributed* among all *unmarked* pages.

When page requested:

- If marked: in memory, no eviction
- If unmarked: if currently i unmarked pages, then $\Pr[\text{eviction}] = \Pr[\text{requested page not in memory}] = 1/i$
  - Becomes marked

At beginning of phase $i = N$, at end of phase $i = 1$. Goes down by one every time page gets marked.

$\Longrightarrow$ expected cost in phase at most $\frac{1}{N} + \frac{1}{N-1} + \frac{1}{N-2} + \cdots + \frac{1}{2} + 1 = O(\log N) = O(\log k)$