# Lecture 25: Online Algorithms

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601.433/633 Introduction to Algorithms

# Introduction

Class until now: difficulty was *computational power*

Today: difficulty is *lack of information*

Online:

- Input / data arrives *over time*
- Need to make decisions without knowing future

# Ski Rental Problem

Want to go skiing, but don't know how many times you'll be able to go this year.

Should you rent or buy?

- Renting skis: $50
- Buying skis: $500
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What's the right strategy (for these costs)?

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- $\boldsymbol{ALG = 450 + 500 = 950}$
- $\boldsymbol{OPT = 500}$

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Never more than twice (actually $\mathbf{\frac{19}{10}}$ times) what we should have done!

# Competitive Ratio

**Definition**

The *competitive ratio* of algorithm $\boldsymbol{ALG}$ is the maximum over all inputs/futures $\boldsymbol{\sigma}$ of

$$\frac{\boldsymbol{ALG(\sigma)}}{\boldsymbol{OPT(\sigma)}},$$

where $\boldsymbol{ALG(\sigma)}$ is the cost of $\boldsymbol{ALG}$ on $\boldsymbol{\sigma}$ and $\boldsymbol{OPT(\sigma)}$ is the optimal cost for $\boldsymbol{\sigma}$ (knowing the future).

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So on ski rental problem with previous values, competitive ratio is $\frac{19}{10}$.

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- $ALG = r \cdot \left(\frac{b}{r} - 1\right) + b = b - r + b = 2b - r$
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So for all inputs / futures, $\frac{ALG}{OPT} \leq 2 - \frac{r}{b}$

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$$\frac{ALG}{OPT} = \frac{xr+b}{(x+1)r} = \frac{xr+b}{xr+r} = 1 + \frac{b-r}{xr+r}$$

$$\geq 1 + \frac{b-r}{(\frac{b}{r}-1)r + r} = 1 + \frac{b-r}{b} = 2 - \frac{r}{b}$$

# Elevator Problem

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# Extensions: Randomization

What if our algorithm is allowed to be randomized?

- Choose our buying time from some distribution $\boldsymbol{p}$ over days (*purchase distribution*)
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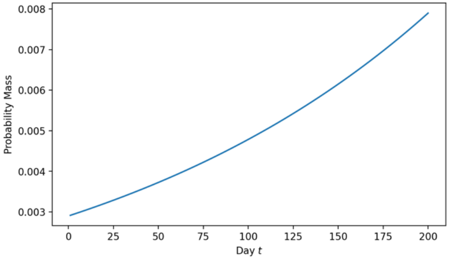
*If we set* $\boldsymbol{p_t} = \left(\frac{b-1}{b}\right)^{b-t} \frac{1}{b\left(1-\left(1-\frac{1}{b}\right)^{b}\right)}$ *for all* $\boldsymbol{t \le b}$*, then*

- $\frac{E[ALG(\sigma)]}{OPT(\sigma)} \le \frac{e}{e-1} \approx \mathbf{1.58}$ *for all* $\boldsymbol{\sigma}$*, and*
- *this is optimal.*



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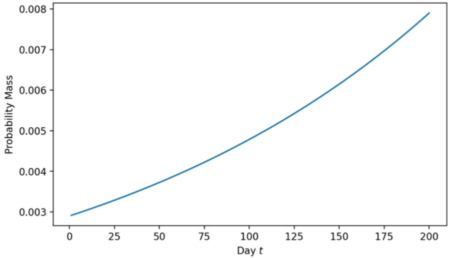


Good in expectation, but what about probability of being bad (e.g., worse than BLTN)?

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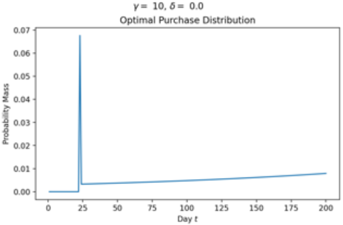


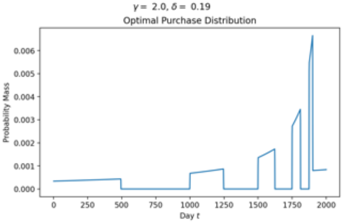


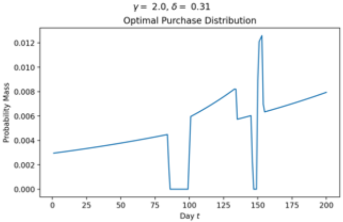

# Paging

Classical problem in computer systems/theory

- Disk (slow) with $N$ pages
- Memory (fast) with room for $k < N$ pages
- If OS/application requests a page not in memory: "page fault"
  - Need to bring requested page into memory, *evict* a page from memory (if currently full)
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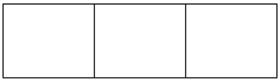
**Example:** $k = 3$. Requests: $\mathbf{1, 2, 3, 2, 4, 3, 4, 1, 2, 3, 4}$



(Convention: initial page faults to fill table don’t count: only pay when we *evict* a page)

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Is this a good algorithm? What’s the competitive ratio? Cost $= \#$ evictions.

- $k = 3$, $N = 4$
- Requests: $1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \dots$

So LRU has competitive ratio $\approx k$

- LRU evicts every time, $OPT$ evicts $1$ out of every $k$ times.

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$$\implies \frac{\boldsymbol{ALG}}{\boldsymbol{OPT}} \geq \boldsymbol{k}$$

# Marking Algorithm

Get around lower bound by using *randomization*

- Lower bound argument doesn't apply because can't set request sequence to ask for whatever's not in memory, since that involved randomness! (Oblivious adversary)

# Marking Algorithm

Get around lower bound by using *randomization*

- Lower bound argument doesn't apply because can't set request sequence to ask for whatever's not in memory, since that involved randomness! (Oblivious adversary)

Assume memory initially $1, 2, \ldots, k$.
Set all pages in memory to be "unmarked"

When page requested:

- If already in memory, "mark" it
- If not in memory:
  - If all pages in memory "marked", unmark all
  - Choose an *unmarked* page uniformly at random to evict
  - Bring in new page, mark it

# Marking Analysis

### Theorem

*Expected competitive ratio at most* $\boldsymbol{O(\log k)}$*:*

$$\frac{\boldsymbol{E[ALG(\sigma)]}}{\boldsymbol{OPT(\sigma)}} \leq \boldsymbol{O(\log k)} \text{ for all request sequences } \boldsymbol{\sigma}.$$

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*Phase*: time between "unmark all" events.

In each phase:

- $OPT \geq 1$, since all $N$ pages requested

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At beginning of phase $\boldsymbol{i} = \boldsymbol{N}$, at end of phase $\boldsymbol{i} = \mathbf{1}$. Goes down by one every time page gets marked.

$\Longrightarrow$ expected cost in phase at most $\frac{1}{N} + \frac{1}{N-1} + \frac{1}{N-2} + \cdots + \frac{1}{2} + 1 = O(\log N) = O(\log k)$