Lecture 3: Probabilistic Analysis, Randomized Quicksort

Michael Dinitz

September 3, 2024
601.433/633 Introduction to Algorithms

Introduction: Sorting

- Sorting: given array of comparable elements, put them in sorted order
- Popular topic to cover in Algorithms courses
- This course:
  - I assume you know the basics (mergesort, quicksort, insertion sort, selection sort, bubble sort, etc.) from Data Structures
  - Today: more advanced sorting (randomized quicksort)
  - Next week: Sorting lower bound and ways around it.

Randomized Algorithms and Probabilistic Analysis

First lecture: "Average-case" problematic.

- What is the "average case"?
- Want to design algorithms that work in *all* applications.

Randomized Algorithms and Probabilistic Analysis

First lecture: "Average-case" problematic.

- What is the "average case"?
- Want to design algorithms that work in *all* applications.

Instead of assuming random distribution over inputs (average-case analysis, machine learning), add randomization *inside* algorithm!

- Still assume worst-case inputs, give bound on worst-case *expected* running time.

Randomized Algorithms and Probabilistic Analysis

First lecture: "Average-case" problematic.

- What is the "average case"?
- Want to design algorithms that work in *all* applications.

Instead of assuming random distribution over inputs (average-case analysis, machine learning), add randomization *inside* algorithm!

- Still assume worst-case inputs, give bound on worst-case *expected* running time.

Many Fall semesters: 601.434/634 Randomized and Big Data Algorithms. Great class!

Randomized Algorithms and Probabilistic Analysis

First lecture: "Average-case" problematic.

- What is the "average case"?
- Want to design algorithms that work in *all* applications.

Instead of assuming random distribution over inputs (average-case analysis, machine learning), add randomization *inside* algorithm!

- Still assume worst-case inputs, give bound on worst-case *expected* running time.

Many Fall semesters: 601.434/634 Randomized and Big Data Algorithms. Great class!

Today: adding randomness into quicksort.

Quicksort Basics (Review)

Input: array A of length n.

Quicksort Basics (Review)

Input: array A of length n.

Algorithm:

1. If $n = 0$ or 1, return A (already sorted)
2. Pick some element p as the *pivot*
3. Compare every element of A to p. Let L be the elements less than p, let G be the elements larger than p. Create array $[L, p, G]$
4. Recursively sort L and G.

Quicksort Basics (Review)

Input: array A of length n.

Algorithm:

1. If $n = 0$ or 1, return A (already sorted)
2. Pick some element p as the *pivot*
3. Compare every element of A to p. Let L be the elements less than p, let G be the elements larger than p. Create array $[L, p, G]$
4. Recursively sort L and G.

Not fully specified: how to choose p?

- Traditionally: some simple deterministic choice (first element, last element, etc.)
- Next lecture: better deterministic choice (not very practical)
- Now: first element

Quicksort Analysis

Upper bound:

If p picked as pivot in step 2, then in correct place after step **3**

Quicksort Analysis

Upper bound:

If p picked as pivot in step 2, then in correct place after step $\mathbf{3}$

$\implies$ step $\mathbf{2}$ and $\mathbf{3}$ executed at most n times.

Quicksort Analysis

Upper bound:
If $\boldsymbol{p}$ picked as pivot in step 2, then in correct place after step $\mathbf{3}$
$\Longrightarrow$ step $\mathbf{2}$ and $\mathbf{3}$ executed at most $\boldsymbol{n}$ times.

Step $\mathbf{3}$ takes time $\boldsymbol{O(n)}$ (compare every element to pivot)

Quicksort Analysis

Upper bound:
If p picked as pivot in step 2, then in correct place after step **3**
$\implies$ step **2** and **3** executed at most n times.

Step **3** takes time $O(n)$ (compare every element to pivot)
$\implies$ total time at most $O(n^2)$

Quicksort Analysis

Upper bound:

If $\boldsymbol{p}$ picked as pivot in step 2, then in correct place after step $\mathbf{3}$
$\implies$ step $\mathbf{2}$ and $\mathbf{3}$ executed at most $\boldsymbol{n}$ times.

Step $\mathbf{3}$ takes time $\boldsymbol{O(n)}$ (compare every element to pivot)
$\implies$ total time at most $\boldsymbol{O(n^2)}$

Lower Bound:

Suppose $\boldsymbol{A}$ already sorted.

Quicksort Analysis

Upper bound:
If p picked as pivot in step 2, then in correct place after step 3
$\implies$ step 2 and 3 executed at most n times.

Step 3 takes time $O(n)$ (compare every element to pivot)
$\implies$ total time at most $O(n^2)$

Lower Bound:
Suppose A already sorted.
$\implies p = A[0]$ is smallest element

Quicksort Analysis

Upper bound:

If p picked as pivot in step 2, then in correct place after step $\mathbf{3}$

$\implies$ step $\mathbf{2}$ and $\mathbf{3}$ executed at most n times.

Step $\mathbf{3}$ takes time $O(n)$ (compare every element to pivot)

$\implies$ total time at most $O(n^2)$

Lower Bound:

Suppose A already sorted.

$\implies p = A[0]$ is smallest element $\implies L = \emptyset$ and $G = A[1..n-1]$

Quicksort Analysis

Upper bound:
If p picked as pivot in step 2, then in correct place after step 3
$\implies$ step 2 and 3 executed at most n times.

Step 3 takes time $O(n)$ (compare every element to pivot)
$\implies$ total time at most $O(n^2)$

Lower Bound:
Suppose A already sorted.
$\implies$ $p = A[0]$ is smallest element $\implies$ $L = \varnothing$ and $G = A[1..n-1]$
$\implies$ in one call to quicksort, do $\Omega(n)$ work to compare everything to p, then recurse on array of size $n-1$

Quicksort Analysis

Upper bound:
If p picked as pivot in step 2, then in correct place after step 3
$\implies$ step 2 and 3 executed at most n times.

Step 3 takes time $O(n)$ (compare every element to pivot)
$\implies$ total time at most $O(n^2)$

Lower Bound:
Suppose A already sorted.
$\implies$ $p = A[0]$ is smallest element $\implies$ $L = \varnothing$ and $G = A[1..n-1]$
$\implies$ in one call to quicksort, do $\Omega(n)$ work to compare everything to p, then recurse on array of size $n-1$
$\implies$ running time is $T(n) = T(n-1) + cn$

Quicksort Analysis

Upper bound:

If p picked as pivot in step 2, then in correct place after step $\mathbf{3}$

$\implies$ step $\mathbf{2}$ and $\mathbf{3}$ executed at most n times.

Step $\mathbf{3}$ takes time $O(n)$ (compare every element to pivot)

$\implies$ total time at most $O(n^2)$

Lower Bound:

Suppose A already sorted.

$\implies p = A[0]$ is smallest element $\implies L = \varnothing$ and $G = A[1..n-1]$

$\implies$ in one call to quicksort, do $\Omega(n)$ work to compare everything to p, then recurse on array of size $n - 1$

$\implies$ running time is $T(n) = T(n-1) + cn \implies T(n) = \Theta(n^2)$

Randomized Quicksort

Randomized Quicksort: pick $\boldsymbol{p}$ *uniformly at random* from $\boldsymbol{A}$.

Today: prove that *expected* running time at most $\boldsymbol{O(n \log n)}$ for *every* input $\boldsymbol{A}$.

Randomized Quicksort

Randomized Quicksort: pick $\boldsymbol{p}$ *uniformly at random* from $\boldsymbol{A}$.

Today: prove that *expected* running time at most $\boldsymbol{O(n \log n)}$ for *every* input $\boldsymbol{A}$.

- Better than an average-case bound: holds for every single input!
- Maybe in one application inputs tend to be pretty well-sorted: original deterministic quicksort bad, this still good!

Randomized Quicksort

Randomized Quicksort: pick $\boldsymbol{p}$ *uniformly at random* from $\boldsymbol{A}$.

Today: prove that *expected* running time at most $\boldsymbol{O(n \log n)}$ for *every* input $\boldsymbol{A}$.

- Better than an average-case bound: holds for every single input!
- Maybe in one application inputs tend to be pretty well-sorted: original deterministic quicksort bad, this still good!
- Today only expectation. Can be more clever to get high probability bounds.

Randomized Quicksort

Randomized Quicksort: pick $\boldsymbol{p}$ *uniformly at random* from $\boldsymbol{A}$.

Today: prove that *expected* running time at most $\boldsymbol{O(n \log n)}$ for *every* input $\boldsymbol{A}$.

- Better than an average-case bound: holds for every single input!
- Maybe in one application inputs tend to be pretty well-sorted: original deterministic quicksort bad, this still good!
- Today only expectation. Can be more clever to get high probability bounds.

Before doing analysis, quick review of basic probability theory.

Probability Basics I

Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

Probability Basics I

Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

$\mathbf{\Omega}$: Sample space. Set of all possible outcomes.

Probability Basics I

Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

$\boldsymbol{\Omega}$: Sample space. Set of all possible outcomes.

- Roll two dice. $\boldsymbol{\Omega} =$

Probability Basics I

Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

$\boldsymbol{\Omega}$: Sample space. Set of all possible outcomes.

- Roll two dice. $\boldsymbol{\Omega = \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\}}$.

Probability Basics I

Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

Ω: Sample space. Set of all possible outcomes.

- Roll two dice. $\Omega = \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\}$. *Not* $\{2, 3, \ldots, 12\}$

Probability Basics I

Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

$\boldsymbol{\Omega}$: Sample space. Set of all possible outcomes.

- Roll two dice. $\boldsymbol{\Omega} = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{6}\} \times \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{6}\}$. *Not* $\{\mathbf{2}, \mathbf{3}, \dots, \mathbf{12}\}$

Event: subset of $\boldsymbol{\Omega}$

Probability Basics I

Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

Ω: Sample space. Set of all possible outcomes.

- Roll two dice. $\Omega = \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\}$. *Not* $\{2, 3, \ldots, 12\}$

Event: subset of Ω

- "Event that first die is 3": $\{(3, x) : x \in \{1, 2, \ldots, 6\}\}$
- "Event that dice add up to 7 or 11": $\{(x, y) \in \Omega : (x + y = 7) \text{ or } (x + y = 11)\}$

Probability Basics I

Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

Ω: Sample space. Set of all possible outcomes.

- Roll two dice. $\Omega = \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\}$. *Not* $\{2, 3, \ldots, 12\}$

Event: subset of Ω

- "Event that first die is 3": $\{(3, x) : x \in \{1, 2, \ldots, 6\}\}$
- "Event that dice add up to 7 or 11": $\{(x, y) \in \Omega : (x + y = 7) \text{ or } (x + y = 11)\}$

Random Variable: $X : \Omega \to \mathbb{R}$

- X_1: value of first die. $X_1(x, y) = x$
- X_2: value of second die. $X_2(x, y) = y$
- $X = X_1 + X_2$: sum of the dice. $X(x, y) = x + y = X_1(x, y) + X_2(x, y)$

Probability Basics I

Only semi-formal here. Look at CLRS Chapter 5 and Appendix C, take Introduction to Probability

Ω: Sample space. Set of all possible outcomes.

- Roll two dice. $\Omega = \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\}$. *Not* $\{2, 3, \ldots, 12\}$

Event: subset of Ω

- "Event that first die is 3": $\{(3, x) : x \in \{1, 2, \ldots, 6\}\}$
- "Event that dice add up to 7 or 11": $\{(x, y) \in \Omega : (x + y = 7) \text{ or } (x + y = 11)\}$

Random Variable: $X : \Omega \to \mathbb{R}$

- X_1: value of first die. $X_1(x, y) = x$
- X_2: value of second die. $X_2(x, y) = y$
- $X = X_1 + X_2$: sum of the dice. $X(x, y) = x + y = X_1(x, y) + X_2(x, y)$

Random variables super important! Running time of randomized quicksort is a random variable.

Probability Basics II

Want to define probabilities. Should use measure theory. Won't.

Probability Basics II

Want to define probabilities. Should use measure theory. Won't.

For each $e \in \Omega$ let $Pr[e]$ be probability of e (probability distribution)

- $Pr[e] \geq 0$ for all $e \in \Omega$, and $\sum_{e \in \Omega} Pr[e] = 1$
- Probability of an event A is $Pr[A] = \sum_{e \in A} Pr[e]$

Probability Basics II

Want to define probabilities. Should use measure theory. Won't.

For each $e \in \Omega$ let $Pr[e]$ be probability of e (probability distribution)

- $Pr[e] \geq 0$ for all $e \in \Omega$, and $\sum_{e \in \Omega} Pr[e] = 1$
- Probability of an event A is $Pr[A] = \sum_{e \in A} Pr[e]$

Conditional probability: if A and B are events:

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]} = \frac{\sum_{e \in A \cap B} Pr[e]}{\sum_{e \in A} Pr[e]}$$

Probability Basics III: Expectations

Expectation of a random variable:

$$\boldsymbol{E}[\boldsymbol{X}] = \sum_{e \in \Omega} \boldsymbol{X}(e) \boldsymbol{Pr}[e]$$

"Average" of the random variable according to probability distribution

Probability Basics III: Expectations

Expectation of a random variable:

$$E[X] = \sum_{e \in \Omega} X(e) Pr[e]$$

"Average" of the random variable according to probability distribution

Can be useful to rearrange terms to get different equation:

$$E[X] = \sum_{e \in \Omega} X(e) Pr[e] = \sum_{y \in \mathbb{R}} \sum_{e \in \Omega : X(e) = y} y \cdot Pr[e] = \sum_{y \in \mathbb{R}} y \cdot Pr[X = y]$$

Probability Basics III: Expectations

Expectation of a random variable:

$$E[X] = \sum_{e \in \Omega} X(e) Pr[e]$$

"Average" of the random variable according to probability distribution

Can be useful to rearrange terms to get different equation:

$$E[X] = \sum_{e \in \Omega} X(e) Pr[e] = \sum_{y \in \mathbb{R}} \sum_{e \in \Omega : X(e) = y} y \cdot Pr[e] = \sum_{y \in \mathbb{R}} y \cdot Pr[X = y]$$

Conditional Expectation: A an event, X a random variable.

$$E[X|A] = \frac{1}{Pr[A]} \sum_{e \in A} X(e) Pr[e]$$

Linearity of Expectations

Amazing feature of expectations: linearity!

Theorem

For any two random variables $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, and any constants* α *and* β*:*

$$\boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] = \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}]$$

Linearity of Expectations

Amazing feature of expectations: linearity!

Theorem

For any two random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$, and any constants α and β:
$\boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] = \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}]$

Consider rolling two dice. Let $\boldsymbol{X}$ be sum. What is $\boldsymbol{E}[\boldsymbol{X}]$?

- $\boldsymbol{E}[\boldsymbol{X}] = \sum_{e \in \Omega} \boldsymbol{X}(e) \boldsymbol{Pr}[e]$. 36 term sum!
- $\boldsymbol{E}[\boldsymbol{X}] = \sum_{y \in \mathbb{R}} y \cdot \boldsymbol{Pr}[\boldsymbol{X} = y]$. What is $\boldsymbol{Pr}[\boldsymbol{X} = 2]$, $\boldsymbol{Pr}[\boldsymbol{X} = 3]$, ...?

Linearity of Expectations

Amazing feature of expectations: linearity!

Theorem

For any two random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$, and any constants α and β:
$\boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] = \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}]$

Consider rolling two dice. Let $\boldsymbol{X}$ be sum. What is $\boldsymbol{E}[\boldsymbol{X}]$?

- $\boldsymbol{E}[\boldsymbol{X}] = \sum_{e \in \Omega} \boldsymbol{X}(e) \boldsymbol{Pr}[e]$. 36 term sum!
- $\boldsymbol{E}[\boldsymbol{X}] = \sum_{y \in \mathbb{R}} y \cdot \boldsymbol{Pr}[\boldsymbol{X} = y]$. What is $\boldsymbol{Pr}[\boldsymbol{X} = 2]$, $\boldsymbol{Pr}[\boldsymbol{X} = 3]$, ...?

Instead: $\boldsymbol{X} = \boldsymbol{X}_1 + \boldsymbol{X}_2$. So $\boldsymbol{E}[\boldsymbol{X}] = \boldsymbol{E}[\boldsymbol{X}_1 + \boldsymbol{X}_2] = \boldsymbol{E}[\boldsymbol{X}_1] + \boldsymbol{E}[\boldsymbol{X}_2]$

Linearity of Expectations

Amazing feature of expectations: linearity!

Theorem

For any two random variables X and Y, and any constants α and β:
$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$

Consider rolling two dice. Let X be sum. What is $E[X]$?

- $E[X] = \sum_{e \in \Omega} X(e) Pr[e]$. 36 term sum!
- $E[X] = \sum_{y \in \mathbb{R}} y \cdot Pr[X = y]$. What is $Pr[X = 2]$, $Pr[X = 3]$, ...?

Instead: $X = X_1 + X_2$. So $E[X] = E[X_1 + X_2] = E[X_1] + E[X_2]$

$$E[X_1] = E[X_2] = \sum_{y=1}^{6} \frac{1}{6} y = \frac{21}{6} = 3.5$$

Linearity of Expectations

Amazing feature of expectations: linearity!

Theorem

For any two random variables X and Y, and any constants α and β:
$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$

Consider rolling two dice. Let X be sum. What is $E[X]$?

- $E[X] = \sum_{e \in \Omega} X(e) Pr[e]$. 36 term sum!
- $E[X] = \sum_{y \in \mathbb{R}} y \cdot Pr[X = y]$. What is $Pr[X = 2]$, $Pr[X = 3]$, ...?

Instead: $X = X_1 + X_2$. So $E[X] = E[X_1 + X_2] = E[X_1] + E[X_2]$

$$E[X_1] = E[X_2] = \sum_{y=1}^{6} \frac{1}{6} y = \frac{21}{6} = 3.5$$

$\implies E[X] = 3.5 + 3.5 = 7$

Linearity of Expectations: Proof

Theorem

For any two random variables $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, and any constants* α *and* β*:*
$\boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] = \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}]$

Proof.

$$\boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] = \sum_{e \in \Omega} \boldsymbol{Pr}[e] \left(\alpha \boldsymbol{X}(e) + \beta \boldsymbol{Y}(e)\right)$$

Linearity of Expectations: Proof

Theorem

For any two random variables X *and* Y, *and any constants* α *and* β:
$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$

Proof.

$$E[\alpha X + \beta Y] = \sum_{e \in \Omega} Pr[e]\left(\alpha X(e) + \beta Y(e)\right)$$
$$= \alpha \sum_{e \in \Omega} Pr[e] X(e) + \beta \sum_{e \in \Omega} Pr[e] X(e)$$

Linearity of Expectations: Proof

Theorem

For any two random variables $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, and any constants* α *and* β*:*
$\boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] = \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}]$

Proof.

$$\begin{aligned} \boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] &= \sum_{e \in \Omega} \boldsymbol{Pr}[e] \left(\alpha \boldsymbol{X}(e) + \beta \boldsymbol{Y}(e)\right) \\ &= \alpha \sum_{e \in \Omega} \boldsymbol{Pr}[e] \boldsymbol{X}(e) + \beta \sum_{e \in \Omega} \boldsymbol{Pr}[e] \boldsymbol{X}(e) \\ &= \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}] \end{aligned}$$

Linearity of Expectations: Proof

Theorem

For any two random variables $\boldsymbol{X}$ *and* $\boldsymbol{Y}$, *and any constants* α *and* β:
$\boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] = \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}]$

Proof.

$$\begin{aligned} \boldsymbol{E}[\alpha \boldsymbol{X} + \beta \boldsymbol{Y}] &= \sum_{e \in \Omega} \boldsymbol{Pr}[e] (\alpha \boldsymbol{X}(e) + \beta \boldsymbol{Y}(e)) \\ &= \alpha \sum_{e \in \Omega} \boldsymbol{Pr}[e] \boldsymbol{X}(e) + \beta \sum_{e \in \Omega} \boldsymbol{Pr}[e] \boldsymbol{X}(e) \\ &= \alpha \boldsymbol{E}[\boldsymbol{X}] + \beta \boldsymbol{E}[\boldsymbol{Y}] \end{aligned}$$

Holds no matter how correlated $\boldsymbol{X}$ and $\boldsymbol{Y}$ are!

Randomized Quicksort I

Theorem

The expected running time of randomized quicksort is at most $O(n \log n)$.

Randomized Quicksort I

Theorem

The expected running time of randomized quicksort is at most $O(n \log n)$.

Assume for simplicity all elements distinct. Running time $= \Theta(\#$ of comparisons$)$

Randomized Quicksort I

Theorem

The expected running time of randomized quicksort is at most $O(n \log n)$.

Assume for simplicity all elements distinct. Running time $= \Theta(\#$ of comparisons$)$

Definitions:

- $X = \#$ of comparisons (random variable)
- $e_i = i$'th smallest element (for $i \in \{1, \ldots, n\}$)
- X_{ij} random variable for all $i, j \in \{1, \ldots, n\}$ with $i < j$:

$$X_{ij} = \begin{cases} 1 & \text{if algorithm compares } e_i \text{ and } e_j \text{ at any point in time} \\ 0 & \text{otherwise} \end{cases}$$

Randomized Quicksort II

Algorithm never compares the same two elements more than once $\implies \boldsymbol{X} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{X_{ij}}$

Randomized Quicksort II

Algorithm never compares the same two elements more than once $\implies X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$

$$E[X] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

Randomized Quicksort II

Algorithm never compares the same two elements more than once $\implies \boldsymbol{X} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{X_{ij}}$

$$\boldsymbol{E}[\boldsymbol{X}] = \boldsymbol{E}\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{X_{ij}}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{E}[\boldsymbol{X_{ij}}]$$

So just need to understand $\boldsymbol{E}[\boldsymbol{X_{ij}}]$

Randomized Quicksort II

Algorithm never compares the same two elements more than once $\implies \boldsymbol{X} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{X_{ij}}$

$$\boldsymbol{E}[\boldsymbol{X}] = \boldsymbol{E}\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{X_{ij}}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{E}[\boldsymbol{X_{ij}}]$$

So just need to understand $\boldsymbol{E}[\boldsymbol{X_{ij}}]$

Simple cases:

Randomized Quicksort II

Algorithm never compares the same two elements more than once $\implies \boldsymbol{X} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{X_{ij}}$

$$\boldsymbol{E}[\boldsymbol{X}] = \boldsymbol{E}\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{X_{ij}}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{E}[\boldsymbol{X_{ij}}]$$

So just need to understand $\boldsymbol{E}[\boldsymbol{X_{ij}}]$

Simple cases:

- $\boldsymbol{j} = \boldsymbol{i} + \mathbf{1}$:

Randomized Quicksort II

Algorithm never compares the same two elements more than once $\implies \boldsymbol{X} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{X_{ij}}$

$$\boldsymbol{E}[\boldsymbol{X}] = \boldsymbol{E}\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{X_{ij}}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \boldsymbol{E}[\boldsymbol{X_{ij}}]$$

So just need to understand $\boldsymbol{E}[\boldsymbol{X_{ij}}]$

Simple cases:

- $\boldsymbol{j} = \boldsymbol{i} + 1$: $\boldsymbol{X_{ij}} = 1$ no matter what, so $\boldsymbol{E}[\boldsymbol{X_{ij}}] = 1$

Randomized Quicksort II

Algorithm never compares the same two elements more than once $\implies X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$

$$E[X] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

So just need to understand $E[X_{ij}]$

Simple cases:

- $j = i + 1$: $X_{ij} = 1$ no matter what, so $E[X_{ij}] = 1$
- $i = 1, j = n$:

Randomized Quicksort II

Algorithm never compares the same two elements more than once $\implies \boldsymbol{X} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$

$$E[X] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

So just need to understand $E[X_{ij}]$

Simple cases:

- $j = i + 1$: $X_{ij} = 1$ no matter what, so $E[X_{ij}] = 1$
- $i = 1, j = n$: e_1 and e_n compared if and only if first pivot chosen is e_1 or e_n
  $\implies E[X_{1n}] = \frac{2}{n}$

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$:

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$: $X_{ij} = 1$

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$: $X_{ij} = 1$

If $e_i < p < e_j$:

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$: $X_{ij} = 1$

If $e_i < p < e_j$: $X_{ij} = 0$

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$: $X_{ij} = 1$

If $e_i < p < e_j$: $X_{ij} = 0$

If $p < e_i$ or $p > e_j$:

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$: $X_{ij} = 1$

If $e_i < p < e_j$: $X_{ij} = 0$

If $p < e_i$ or $p > e_j$: ? Both e_i, e_j in same recursive call.

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$: $X_{ij} = 1$

If $e_i < p < e_j$: $X_{ij} = 0$

If $p < e_i$ or $p > e_j$: ? Both e_i, e_j in same recursive call.

- Condition on $e_i \le p \le e_j$:

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$: $X_{ij} = 1$

If $e_i < p < e_j$: $X_{ij} = 0$

If $p < e_i$ or $p > e_j$: ? Both e_i, e_j in same recursive call.

- Condition on $e_i \leq p \leq e_j$: $E[X_{ij} \mid e_i \leq p \leq e_j] = \frac{2}{j-i+1}$

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$: $X_{ij} = 1$

If $e_i < p < e_j$: $X_{ij} = 0$

If $p < e_i$ or $p > e_j$: ? Both e_i, e_j in same recursive call.

- Condition on $e_i \leq p \leq e_j$: $E[X_{ij} \mid e_i \leq p \leq e_j] = \frac{2}{j-i+1}$
- Condition on $p \notin [e_i, e_j]$:

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$: $X_{ij} = 1$

If $e_i < p < e_j$: $X_{ij} = 0$

If $p < e_i$ or $p > e_j$: ? Both e_i, e_j in same recursive call.

- Condition on $e_i \le p \le e_j$: $E[X_{ij} \mid e_i \le p \le e_j] = \frac{2}{j-i+1}$
- Condition on $p \notin [e_i, e_j]$: still undetermined

$E[X_{ij}]$: General Case ($i < j$)

If $p = e_i$ or $p = e_j$: $X_{ij} = 1$

If $e_i < p < e_j$: $X_{ij} = 0$

If $p < e_i$ or $p > e_j$: ? Both e_i, e_j in same recursive call.

- Condition on $e_i \le p \le e_j$: $E[X_{ij} \mid e_i \le p \le e_j] = \frac{2}{j-i+1}$
- Condition on $p \notin [e_i, e_j]$: still undetermined

So X_{ij} not determined until $e_i \le p \le e_j$, and when it is determined has $E[X_{ij}] = \frac{2}{j-i+1}$
$\implies E[X_{ij}] = \frac{2}{j-i+1}$

$E[X_{ij}]$: General Case (formally)

Let Y_k be event that the k'th pivot is in $[e_i, e_j]$ and all previous pivots not in $[e_i, e_j]$

$E[X_{ij}]$: General Case (formally)

Let Y_k be event that the k'th pivot is in $[e_i, e_j]$ and all previous pivots not in $[e_i, e_j]$
$\implies$ by definition, the Y_k events are disjoint and partition sample space

$E[X_{ij}]$: General Case (formally)

Let Y_k be event that the k'th pivot is in $[e_i, e_j]$ and all previous pivots not in $[e_i, e_j]$
$\implies$ by definition, the Y_k events are disjoint and partition sample space

Showed that $E[X_{ij}|Y_k] = \frac{2}{j-i+1}$ for all k.

$E[X_{ij}]$: General Case (formally)

Let Y_k be event that the k'th pivot is in $[e_i, e_j]$ and all previous pivots not in $[e_i, e_j]$
$\implies$ by definition, the Y_k events are disjoint and partition sample space

Showed that $E[X_{ij}|Y_k] = \frac{2}{j-i+1}$ for all k.

$$E[X_{ij}] = \sum_{k=1}^{n} E[X_{ij}|Y_k]Pr[Y_k] \qquad (Y_k \text{ disjoint and partition } \Omega)$$

$$= \frac{2}{j-i+1} \sum_{k=1}^{n} Pr[Y_k]$$

$$= \frac{2}{j-i+1}$$

Randomized Quicksort: Final Analysis

Expected running time of randomized quicksort:

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] && \text{(linearity of expectations)} \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= 2 \sum_{i=1}^{n-1} \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-i+1} \right) \\
&\leq 2 \sum_{i=1}^{n-1} H_n && \left(H_n = \sum_{j=1}^{n} \frac{1}{j} \right) \\
&\leq 2nH_n \\
&\leq O(n \log n)
\end{aligned}
$$