Lecture 4: Linear Time Selection/Median

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601.433/633 Introduction to Algorithms

Intro and Problem Definition

Last time: sorting in expected $O(n \log n)$ time (randomized quicksort)

- Should already know (from Data Structures) deterministic $O(n \log n)$ algorithms for sorting (mergesort, heapsort)

Today: two related problems

- Median: Given array A of length n, find the median: $\lceil n/2 \rceil$nd smallest element.
- Selection: Given array A of length n and $k \in [n] = \{1, 2, \ldots, n\}$, find k'th smallest element.

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Can solve both in $O(n \log n)$ time via sorting. Faster?

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- Need to keep track of $n/2$ smallest.
- When scanning, see an element, need to determine if one of k smallest. If yes, remove previous max of the $n/2$ we've been keeping track of.
    - Not easy to do! Foreshadow: would need to use a *heap*. $\Theta(\log n)$-worst case time.

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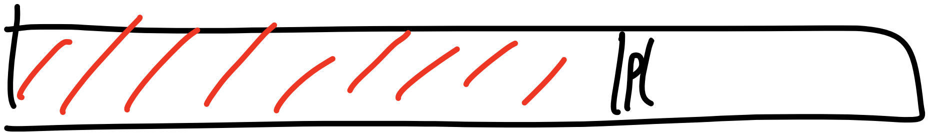
- $\Theta(n \log n)$ worst-case time.

(Randomized) Quickselect

Main idea: (Randomized) Quicksort, but only recurse on side with element we're looking for.



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R-Quickselect($\boldsymbol{A}, \boldsymbol{k}$):

1. If $|\boldsymbol{A}| = \mathbf{1}$, return the element.
2. Pick a pivot element $\boldsymbol{p}$ uniformly at random from $\boldsymbol{A}$.
3. Compare each element of $\boldsymbol{A}$ to $\boldsymbol{p}$, creating subarrays $\boldsymbol{L}$ of elements less than $\boldsymbol{p}$ and $\boldsymbol{G}$ of elements greater than $\boldsymbol{p}$.
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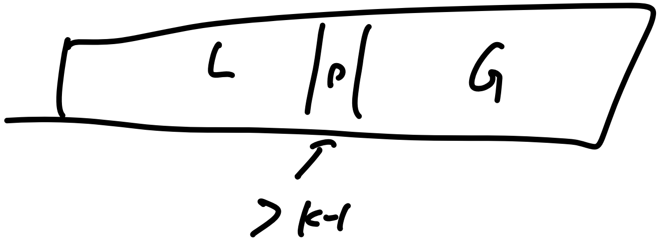

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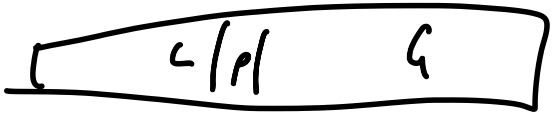

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$k - (|L| + 1)$

Quickselect: Correctness

Sketch here: good exercise to do at home!

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Prove by induction ("loop invariant") that on any call to R-Quickselect($\boldsymbol{X}, \boldsymbol{a}$), the element we're looking for is $\boldsymbol{a}$'th smallest of $\boldsymbol{X}$.

- Base case: first call to R-Quickselect($\boldsymbol{A}, \boldsymbol{k}$). Correct by definition.
- Inductive case: suppose was true for call R-Quickselect($\boldsymbol{Y}, \boldsymbol{b}$).
  - If we return element: correct
  - If we recurse on $\boldsymbol{L}$: correct
  - If we recurse on $\boldsymbol{G}$: correct

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- Random pivot should be "near middle", so splits array "approximately in half".
- $O(\log n)$ recursive calls, but each one on an array of half the size
  $\implies T(n) = T(n/2) + cn \implies O(n)$ time

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$$T(n) \leq (n-1) + \sum_{i=0}^{n-1} \frac{1}{n} T(\max(i, n-i-1))$$

w.p. $\frac{1}{n}$, $|L| = i$ $\forall i \in \{0, 1, \ldots, n-1\}$

$$\leq (n-1) + \sum_{i=0}^{n/2-1} \frac{1}{n} T(n-i-1) + \sum_{i=n/2}^{n-1} \frac{1}{n} T(i) = (n-1) + \frac{2}{n} \sum_{i=n/2}^{n-1} T(i)$$

Quickselect: Running Time II

Want to solve recurrence relation $T(n) \leq (n-1) + \frac{2}{n} \sum_{i=n/2}^{n-1} T(i)$.

Guess and check: $T(n) \leq 4n$.

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$$
\begin{aligned}
T(n) &\le (n-1) + \frac{2}{n}\sum_{i=n/2}^{n-1} 4i = (n-1) + 4\cdot\frac{2}{n}\sum_{i=n/2}^{n-1} i \\
&= (n-1) + 4\cdot\frac{2}{n}\left(\sum_{i=1}^{n-1} i - \sum_{i=1}^{n/2-1} i\right) \\
&= (n-1) + 4\cdot\frac{2}{n}\left(\frac{n(n-1)}{2} - \frac{(n/2)(n/2-1)}{2}\right) \\
&\le (n-1) + 4\cdot\left((n-1) - \frac{n/2-1}{2}\right) \\
&\le (n-1) + 4\left(\frac{3n}{4}\right) \le 4n.
\end{aligned}
$$

Deterministic Version

Intuition:

- Randomization worked because it got us a "reasonably good" pivot.
- Simple deterministic pivot (first element, last element, etc.) bad because might not split array well.
- Deterministically find a pivot that's "close" to the middle?

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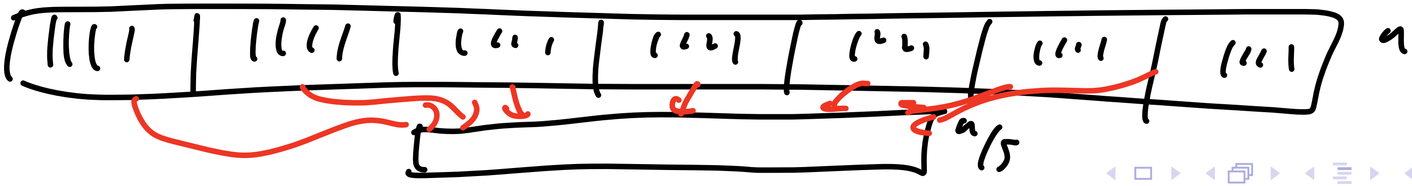
Median-of-medians:

- Split A into $n/5$ groups of 5 elements each.
- Compute median of each group.
- Let p be the median of the $n/5$ medians



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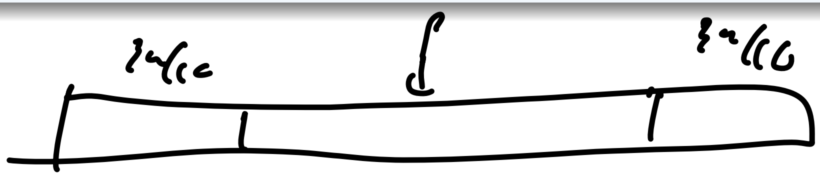
Want to claim: p is a good pivot, and can find p efficiently.

Median-of-Medians is good pivot

Theorem

$|L|$ and $|G|$ are both at most $7n/10$ when p is median of medians.



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Let B be a group (of 5 elements), m median of B:

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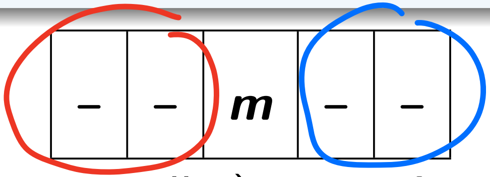

- If $m < p$: at least three elements of B (m and two smaller) are in L

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Let $\mathbf{B}$ be a group (of $\mathbf{5}$ elements), $\mathbf{m}$ median of $\mathbf{B}$:

_	_	$\mathbf{m}$	_	_

- If $\mathbf{m} < \mathbf{p}$: at least three elements of $\mathbf{B}$ ($\mathbf{m}$ and two smaller) are in $\mathbf{L}$
- If $\mathbf{m} > \mathbf{p}$: at least three elements of $\mathbf{B}$ ($\mathbf{m}$ and two larger) are in $\mathbf{G}$

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By definition of p, $n/10$ groups have $m < p$ and $n/10$ have $m > p$

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$$|L| \geq \frac{n}{10} \cdot 3 = \frac{3n}{10} \implies |G| \leq \frac{7n}{10}$$

Median-of-Medians is good pivot

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$$|G| \geq \frac{n}{10} \cdot 3 = \frac{3n}{10} \implies |L| \leq \frac{7n}{10}$$

$|L| \geq \frac{n}{6} \cdot 2 = \frac{n}{3}$

$\Rightarrow |G| \leq \frac{2n}{3}$

$|L| \leq 2n/3$

Finding Median of Medians

Have $n/5$ elements (median of each group). Want to find median.

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Recursion! Use same algorithm on array of medians.

BPFRT

Algorithm due to Blum-Pratt-Floyd-Rivest-Tarjan.

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1. Group $\boldsymbol{A}$ into $\boldsymbol{n/5}$ groups of $\mathbf{5}$, and let $\boldsymbol{A'}$ be an array of size $\boldsymbol{n/5}$ containing the median of each group.
2. Let $\boldsymbol{p} = \text{BPFRT}(\boldsymbol{A'}, \boldsymbol{n/10})$, i.e., recursively find the median $\boldsymbol{p}$ of $\boldsymbol{A'}$ (the median-of-the-medians).

BPFRT

Groups size 3

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1. Group $\boldsymbol{A}$ into $\boldsymbol{n/5}$ groups of $\mathbf{5}$, and let $\boldsymbol{A'}$ be an array of size $\boldsymbol{n/5}$ containing the median of each group. $O(n)$ $O(n)$
2. Let $\boldsymbol{p}$ = BPFRT($\boldsymbol{A'}, \boldsymbol{n/10}$), i.e., recursively find the median $\boldsymbol{p}$ of $\boldsymbol{A'}$ (the median-of-the-medians). $T(n/5)$ $T(n/3)$
3. Split $\boldsymbol{A}$ using $\boldsymbol{p}$ as a pivot into $\boldsymbol{L}$ and $\boldsymbol{G}$. $O(n)$ $O(n)$
4. Recurse on the appropriate piece:
   4.1 if $|\boldsymbol{L}| = \boldsymbol{k} - \mathbf{1}$ then return $\boldsymbol{p}$.
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$T(7n/10)$ $T(2n/3)$

BPFRT Analysis

Let $\boldsymbol{T(n)}$ be (worst-case) running time on $\boldsymbol{A}$ of size $\boldsymbol{n}$.

- Step 1: $\boldsymbol{O(n)}$ time
- Step 2: $\boldsymbol{T(n/5)}$ time
- Step 3: $\boldsymbol{O(n)}$ time
- Step 4: $\boldsymbol{T(7n/10)}$ time

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$T(2n/3) + T(n/3) + cn$

$$T(n) \leq T(7n/10) + T(n/5) + cn$$

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$$T(n) \leq T(7n/10) + T(n/5) + cn$$

Guess $T(n) \leq 10cn$:

$$T(n) \leq 10c(7n/10) + 10c(n/5) + cn = 9cn + cn = 10cn$$

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Let $T(n)$ be time on input of size n.

- BPFRT to find pivot takes $O(n)$ time
- Splitting around pivot takes $O(n)$ time
- Each recursive call takes $T(n/2)$ time

Deterministic $O(n \log n)$ Quicksort

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$$T(n) = 2T(n/2) + cn \implies T(n) = \Theta(n \log n)$$