Lecture 4: Linear Time Selection/Median

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601.433/633 Introduction to Algorithms

Intro and Problem Definition

Last time: sorting in expected $O(n \log n)$ time (randomized quicksort)

- Should already know (from Data Structures) deterministic $O(n \log n)$ algorithms for sorting (mergesort, heapsort)

Today: two related problems

- Median: Given array A of length n, find the median: $\lceil n/2 \rceil$nd smallest element.
- Selection: Given array A of length n and $k \in [n] = \{1, 2, \ldots, n\}$, find k'th smallest element.

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Can solve both in $O(n \log n)$ time via sorting. Faster?

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- Need to keep track of $n/2$ smallest.
- When scanning, see an element, need to determine if one of k smallest. If yes, remove previous max of the $n/2$ we've been keeping track of.
    - Not easy to do! Foreshadow: would need to use a *heap*. $\Theta(\log n)$-worst case time.

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- $\Theta(n \log n)$ worst-case time.

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Main idea: (Randomized) Quicksort, but only recurse on side with element we're looking for.

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1. If $|A| = 1$, return the element.
2. Pick a pivot element p uniformly at random from A.
3. Compare each element of A to p, creating subarrays L of elements less than p and G of elements greater than p.
4. 4.1 If $|L| = k - 1$:

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   4.1 If $|L| = k - 1$: return p.
   4.2 if $|L| > k - 1$: return R-Quickselect(L, k).
   4.3 If $|L| < k - 1$: return R-Quickselect($G, k - |L| - 1$).

Quickselect: Correctness

Sketch here: good exercise to do at home!

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Prove by induction ("loop invariant") that on any call to R-Quickselect($\boldsymbol{X}, \boldsymbol{a}$), the element we're looking for is $\boldsymbol{a}$'th smallest of $\boldsymbol{X}$.

- Base case: first call to R-Quickselect($\boldsymbol{A}, \boldsymbol{k}$). Correct by definition.
- Inductive case: suppose was true for call R-Quickselect($\boldsymbol{Y}, \boldsymbol{b}$).
  - If we return element: correct
  - If we recurse on $\boldsymbol{L}$: correct
  - If we recurse on $\boldsymbol{G}$: correct

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  $\implies T(n) = T(n/2) + cn \implies O(n)$ time

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$$
\begin{aligned}
T(n) &\leq (n-1) + \sum_{i=0}^{n-1} \frac{1}{n} T(\max(i, n-i-1)) \\
&\leq (n-1) + \sum_{i=0}^{n/2-1} \frac{1}{n} T(n-i-1) + \sum_{i=n/2}^{n-1} \frac{1}{n} T(i) = (n-1) + \frac{2}{n} \sum_{i=n/2}^{n-1} T(i)
\end{aligned}
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Quickselect: Running Time II

Want to solve recurrence relation $T(n) \le (n-1) + \frac{2}{n}\sum_{i=n/2}^{n-1} T(i)$.

Guess and check: $T(n) \le 4n$.

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$$\begin{aligned}
T(n) &\leq (n-1) + \frac{2}{n}\sum_{i=n/2}^{n-1} 4i = (n-1) + 4 \cdot \frac{2}{n}\sum_{i=n/2}^{n-1} i \\
&= (n-1) + 4 \cdot \frac{2}{n}\left(\sum_{i=1}^{n-1} i - \sum_{i=1}^{n/2-1} i\right) \\
&= (n-1) + 4 \cdot \frac{2}{n}\left(\frac{n(n-1)}{2} - \frac{(n/2)(n/2-1)}{2}\right) \\
&\leq (n-1) + 4 \cdot \left((n-1) - \frac{n/2-1}{2}\right) \\
&\leq (n-1) + 4\left(\frac{3n}{4}\right) \leq 4n.
\end{aligned}$$

Deterministic Version

Intuition:

- Randomization worked because it got us a “reasonably good” pivot.
- Simple deterministic pivot (first element, last element, etc.) bad because might not split array well.
- Deterministically find a pivot that’s “close” to the middle?

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- Split A into $n/5$ groups of 5 elements each.
- Compute median of each group.
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Want to claim: p is a good pivot, and can find p efficiently.

Median-of-Medians is good pivot

Theorem

$|L|$ *and* $|G|$ *are both at most* $7n/10$ *when* p *is median of medians.*

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$$|G| \geq \frac{n}{10} \cdot 3 = \frac{3n}{10} \implies |L| \leq \frac{7n}{10}$$

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Recursion! Use same algorithm on array of medians.

BPFRT

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BPFRT$(\boldsymbol{A}, \boldsymbol{k})$

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BPFRT(A, k)

1. Group A into $n/5$ groups of 5, and let A' be an array of size $n/5$ containing the median of each group.
2. Let $p = \text{BPFRT}(A', n/10)$, i.e., recursively find the median p of A' (the median-of-the-medians).

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2. Let $\boldsymbol{p} = \text{BPFRT}(\boldsymbol{A'}, \boldsymbol{n/10})$, i.e., recursively find the median $\boldsymbol{p}$ of $\boldsymbol{A'}$ (the median-of-the-medians).
3. Split $\boldsymbol{A}$ using $\boldsymbol{p}$ as a pivot into $\boldsymbol{L}$ and $\boldsymbol{G}$.
4. Recurse on the appropriate piece:
   4.1 if $|\boldsymbol{L}| = \boldsymbol{k} - \boldsymbol{1}$ then return $\boldsymbol{p}$.
   4.2 if $|\boldsymbol{L}| > \boldsymbol{k} - \boldsymbol{1}$ then return BPFRT$(\boldsymbol{L}, \boldsymbol{k})$.
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BPFRT Analysis

Let $T(n)$ be (worst-case) running time on A of size n.

- Step 1: $O(n)$ time
- Step 2: $T(n/5)$ time
- Step 3: $O(n)$ time
- Step 4: $T(7n/10)$ time

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Guess $T(n) \le 10cn$:

$$T(n) \le 10c(7n/10) + 10c(n/5) + cn = 9cn + cn = 10cn$$

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Let $T(n)$ be time on input of size n.

- BPFRT to find pivot takes $O(n)$ time
- Splitting around pivot takes $O(n)$ time
- Each recursive call takes $T(n/2)$ time

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$$T(n) = 2T(n/2) + cn \implies T(n) = \Theta(n \log n)$$