Lecture 6: Balanced Search Trees

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September 12, 2024

601.433/633 Introduction to Algorithms

Announcements

- HW2 due now, HW3 released
- Regrade policy: 72 hours from when grades released
  - Don't abuse this!
  - If too many of your regrade requests do not result in positive changes, will ban you from regrade requests
  - Grading can go down!

Introduction

Today, and next few weeks: data structures.

- Since “Data Structures” a prereq, focus on advanced structures and on interesting analysis

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Today and later: data structures for *dictionaries*

Definition

A *dictionary data structure* is a data structure supporting the following operations:

- **insert(key,object)**: insert the (key, object) pair.
- **lookup(key)**: return the associated object
- **delete(key)**: remove the key and its object from the data structure. We may or may not care about this operation.

Obvious Approaches

Reminder: all running times for *worst case*

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Goal: $O(\log n)$ for both.

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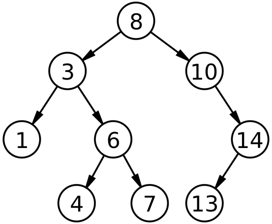
Approach today: search trees

Binary Search Tree Review

Binary search tree:

- All nodes have at most **2** children
- Each node stores (key, object) pair
- All descendants to left have smaller keys
- All descendants to the right have larger keys



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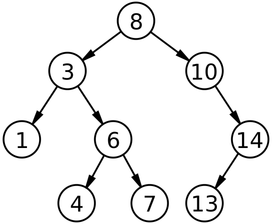

Lookup: follow path from root!

Dictionary Operations in Simple Binary Search Tree

insert($\boldsymbol{x}$):

- If tree empty, put $\boldsymbol{x}$ at root
- Else if $\boldsymbol{x} < \boldsymbol{root.key}$ recursively insert into left child
- Else (if $\boldsymbol{x} > \boldsymbol{root.key}$) recursively insert into right child

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Example: `H O P K I N S`

Simply Binary Search Tree: Analysis

Pluses: easy to implement

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Rest of today:

- B-trees: perfect balance, not binary
- Red-black trees: approximate balance, binary
- Turn out to be related!

B-Trees

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Definition (B-tree with parameter t)

1. Each node has between $t-1$ and $2t-1$ keys in it (except the root has between 1 and $2t-1$ keys). Keys in a node are stored in a sorted array.
2. Each non-leaf has degree (number of children) equal to the number of keys in it plus 1. If v is a node with keys $[a_1, a_2, \ldots, a_k]$ and the children are $[v_1, v_2, \ldots, v_{k+1}]$, then the tree rooted at v_i contains only keys that are at least a_{i-1} and at most a_i (except the the edge cases: the tree rooted at v_1 has keys less than a_1, and the tree rooted at v_{k+1} has keys at least a_k).
3. All leaves are at the same depth.

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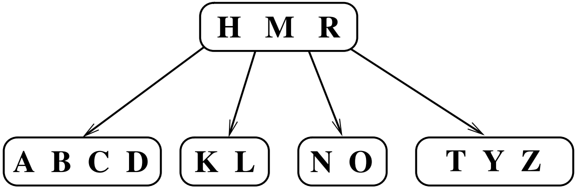
3. All leaves are at the same depth.

When $t = 2$ known as a *2-3-4 tree*, since # children either 2, 3, or 4

B-tree: Example

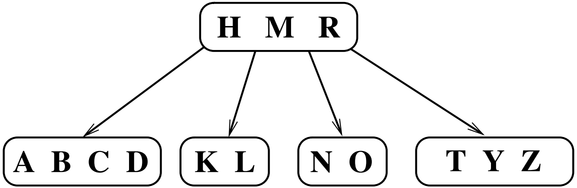
$t = 3$:

- Root has between **1** and **5** keys, non-roots have between **2** and **5** keys
- Non-leaves have between **3** and **6** children (root can have fewer).

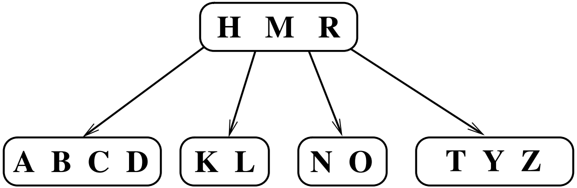


Lookups

Binary search in array at root. Finished if find item, else get pointer to appropriate child, recurse.



Insert(x)



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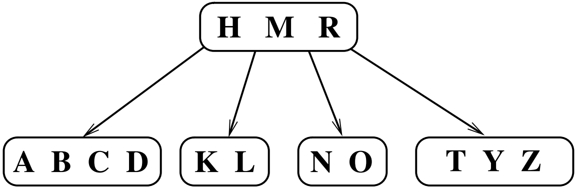

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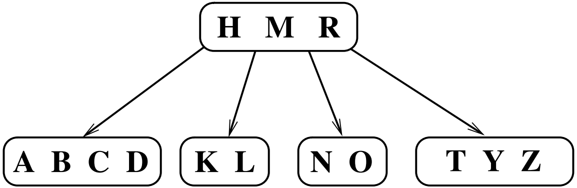

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Split:

- Only used on *full* nodes (nodes with $2t - 1$ keys) whose parents are *not* full.
- Pull median of its keys up to its parent
- Split remaining $2t - 2$ keys into two nodes of $t - 1$ keys each. Reconnect appropriately.

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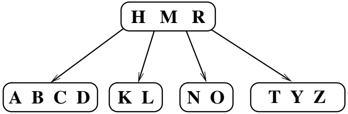

Insert $\boldsymbol{E}, \boldsymbol{F}$ into example.

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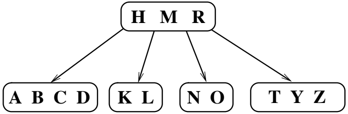

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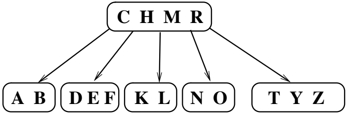

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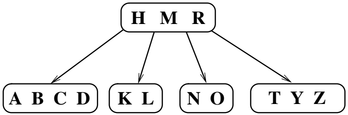

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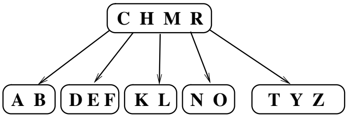
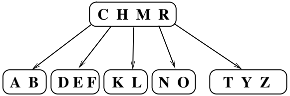


Note: since split *on the way down*, when a node is split, its parent is not full!

Example continued



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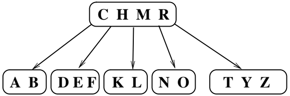

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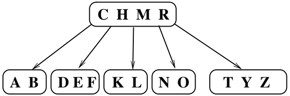

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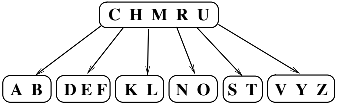

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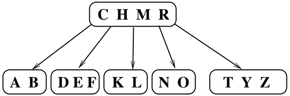

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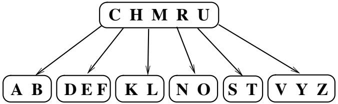

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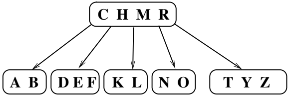

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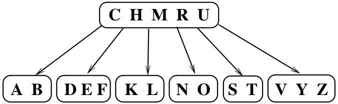

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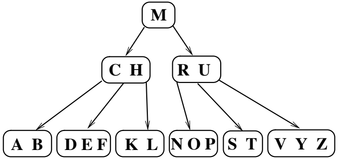

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Induction. Start with a valid B-tree, insert x.

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Second property (correct degrees, subtrees have keys in correct ranges): Hooked nodes up correctly after split. ✓

B-tree running time

Suppose $\boldsymbol{n}$ keys, depth $\boldsymbol{d}$

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- $O(t \log_t n) = O(\frac{t}{\log t} \log n)$ total

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$t = 2$:

- 2-3-4 tree
- Can be implemented as *binary* tree using *red-black trees*

Red-Black Trees

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Most famous: *red-black trees*

- Default in Linux kernel, used to optimize Java HashMap, ...
- Today: Quick overview, connection to 2-3-4 trees.
- *Not* traditional or practical point of view on red-black trees. See book!

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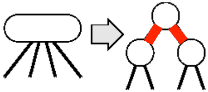

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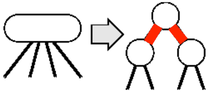

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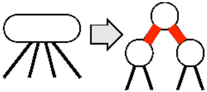

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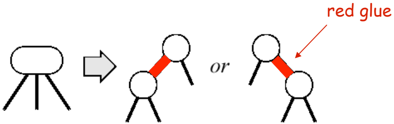
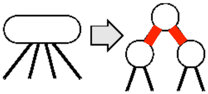
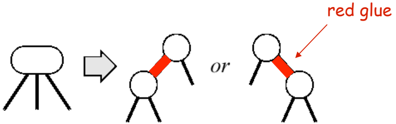


Important Properties





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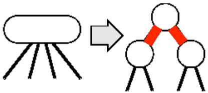

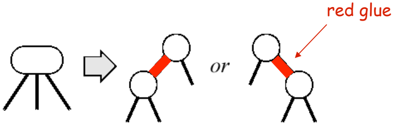

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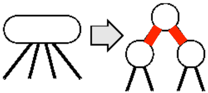

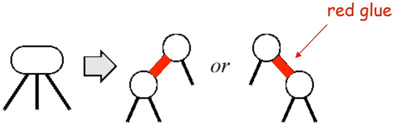

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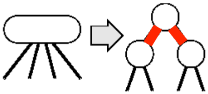

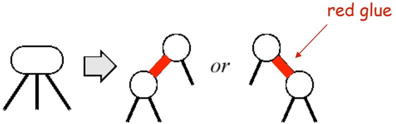

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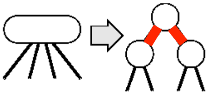

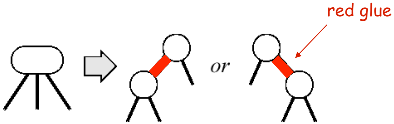

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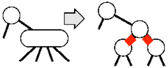

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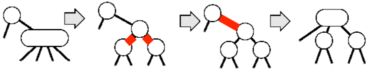

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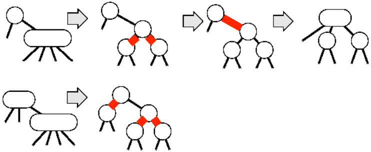

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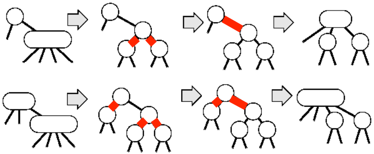

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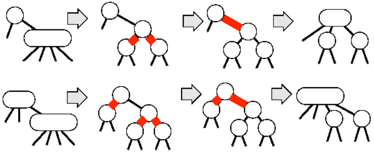
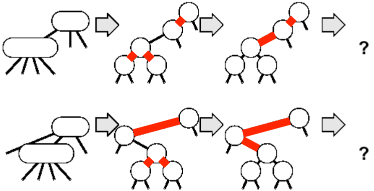


Harder cases:



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Used in many different tree constructions.

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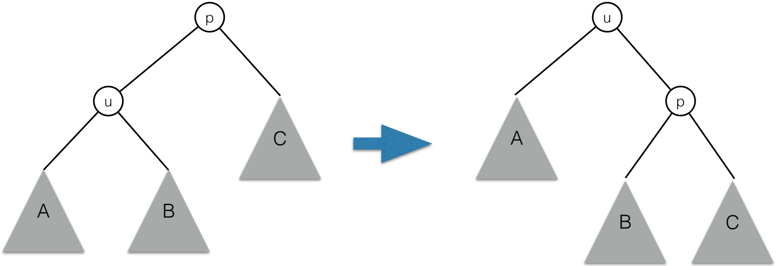


Using Rotations

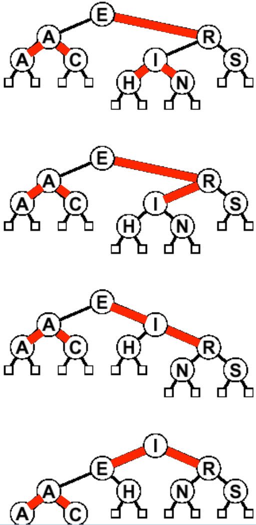
Can use rotations to “fix” hard cases. Example:

inserting G

change colors

right rotate R →

left rotate E →



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Main points:

- Red-Black trees can be thought of as a binary implementation of 2-3-4 trees
- Approximately balanced, so $O(\log n)$ lookup time
- Insert time (basically) same as 2-3-4 tree, so also $O(\log n)$.
- See book for direct approach (not through 2-3-4 trees).