Lecture 7: Amortized Analysis

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601.433/633 Introduction to Algorithms

Introduction

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- Dictionary: insert, insert, insert, lookup, insert, lookup, lookup, ...

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Data structures: *sequence* of operations!

- Dictionary: insert, insert, insert, lookup, insert, lookup, lookup, ...

Last time: analyzed the (worst-case) cost of each operation.
What about (worst-case) cost of *sequence* of operations?

Definition & Example

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- Normal worst-case analysis: $\mathbf{100}$
- Amortized cost: $\mathbf{200/101 \approx 2}$

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The *amortized cost* of a sequence of n operations is the total cost of the sequence divided by n.

"Average cost per operation" (but no randomness!)

Example: 100 operations of cost 1, then 1 operation of cost 100

- Normal worst-case analysis: 100
- Amortized cost: $200/101 \approx 2$

If we care about total time (e.g., using data structure in larger algorithm) then worst-case too pessimistic

Amortized Algorithm

Still want worst-case, but worst-case over *sequences* rather than single operations.

Maybe only possible way to have an expensive operation is to have a bunch of cheap operations: amortized cost always small!

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Definition

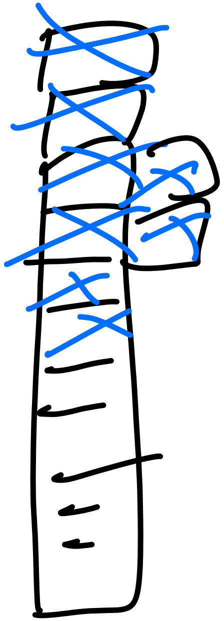
If the amortized cost of *every* sequence of n operations is at most $f(n)$, then the *amortized cost* or *amortized complexity* of the algorithm is at most $f(n)$.

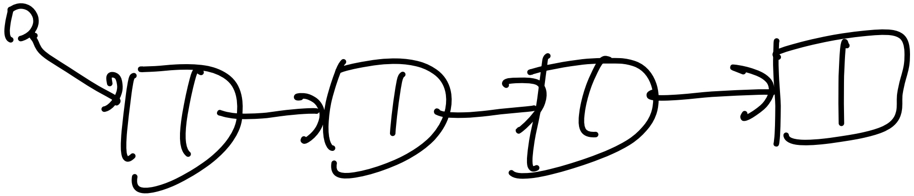
Example: Stack From Array

Stack Using Array

Stack:

- Last In First Out (LIFO)
- Push: add element to stack
- Pop: Remove the most recently added element.





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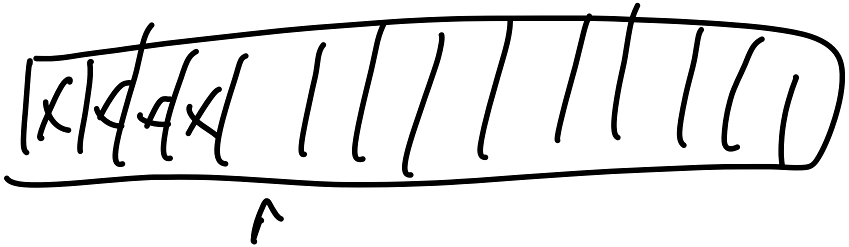

Stack Using Array

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- Last In First Out (LIFO)
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Building a stack with an array `A`:

- Initialize: `top = 0`
- Push(x): `A[top] =` x`; top++`
- Pop: `top--; Return A[top]`

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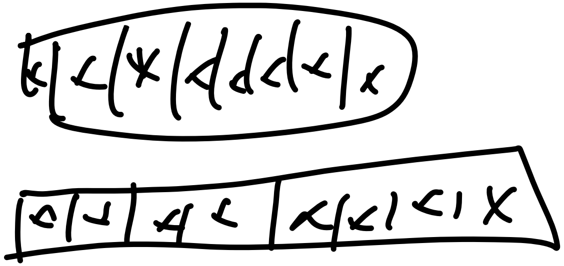
- Cost: free to create new array, each copy costs $\mathbf{1}$
- Worst case: a single Push could cost $\mathbf{\Omega}(n)$!



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New array has size $n+1$:

- Sequence of n Push operations. Total cost: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$.
- Amortized cost: $\Theta(n)$ (same as worst single operation!)

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Consider *any* sequence of n operations.

- Have to double when array has size $2, 4, 8, 16, 32, 64, \ldots, \cancel{\lfloor \log n \rfloor}$ $2^{\lfloor \log n \rfloor}$
- *Total* time spent doubling: at most $\sum_{i=1}^{\lfloor \log n \rfloor} 2^i \leq 2n = \Theta(n)$
- Any operation that doesn't cause a doubling costs $O(1)$
- Total cost at most $O(n) + n \cdot O(1) = O(n)$
- Amortized cost at most $O(1)$

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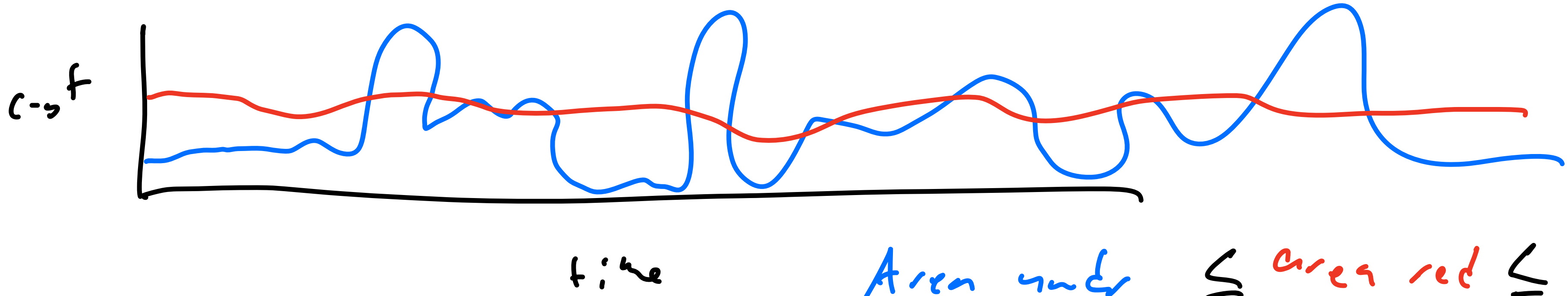
Amortized analysis explains why it's better to double than add 1!

More Complicated Analysis: Piggy Banks and Potentials

Basic Bank: Informal

Can be hard to give good bound directly on total cost.

- Lots of variance: some operations very expensive, some very cheap.
- Idea: “smooth out” the operations.
- “Pay more” for cheap operations, “pay less” for expensive ops.



Area under $\leq$ area red $\leq f(n) \cdot n$

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Use a “bank” to keep track of this

- Cheap operation: add to the bank
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Charge cheap operations more, use extra to pay for expensive operations

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Bank L.

- Initially $L = 0$
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Total cost of sequence:

telescoping sum

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} \left(c'_i + L_{i-1} - L_i\right) = \sum_{i=1}^{n} c'_i + \sum_{i=1}^{n} (L_{i-1} - L_i) = \left(\sum_{i=1}^{n} c'_i\right) + L_0 - L_n$$

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So if $L_0 = 0$ and $L_n \geq 0$ (bank not negative): $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} c_i'$

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So if $L_0 = 0$ and $L_n \geq 0$ (bank not negative): $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} c'_i$

- If $c'_i \leq f(n)$ for all i, then "true" amortized cost $(\sum_{i=1}^{n} c_i)/n$ also at most $f(n)$!

Variants

Multiple banks

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Potential Functions:

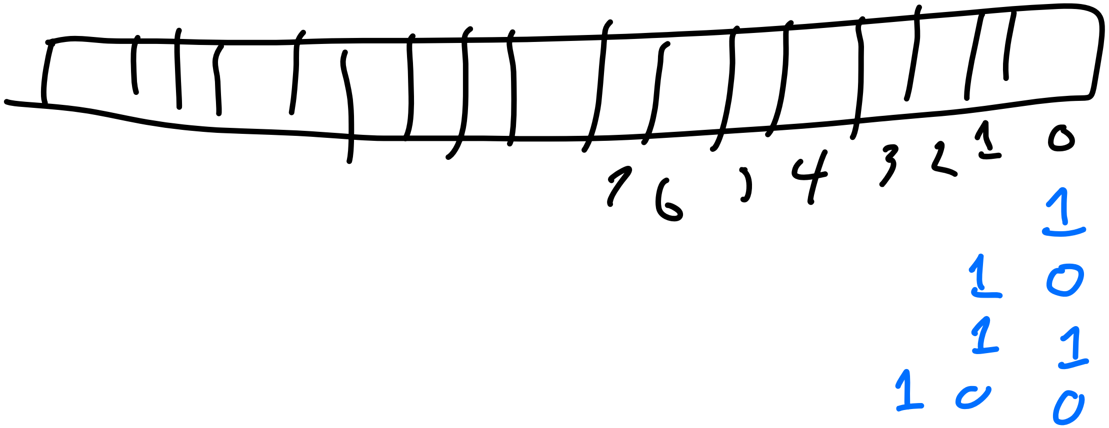
- "Bank analogy": we choose how much to deposit/withdraw.
- New analogy: "potential energy". Function of state of system.
- Rename $\boldsymbol{L}$ to $\boldsymbol{\Phi}$: all previous analysis works same!
- Sometimes easier to think about: just define once at the beginning, instead of for each operation.

Example: Binary Counter

Binary Counter

Super simple setup: binary counter stored in array $\boldsymbol{A}$.

- Least significant bit in $\boldsymbol{A}[\mathbf{0}]$, then $\boldsymbol{A}[\mathbf{1}]$, ...
- Don't worry about length of array (infinite, or long enough)
- Only operation is increment.
- Costs $\mathbf{1}$ to flip any bit.



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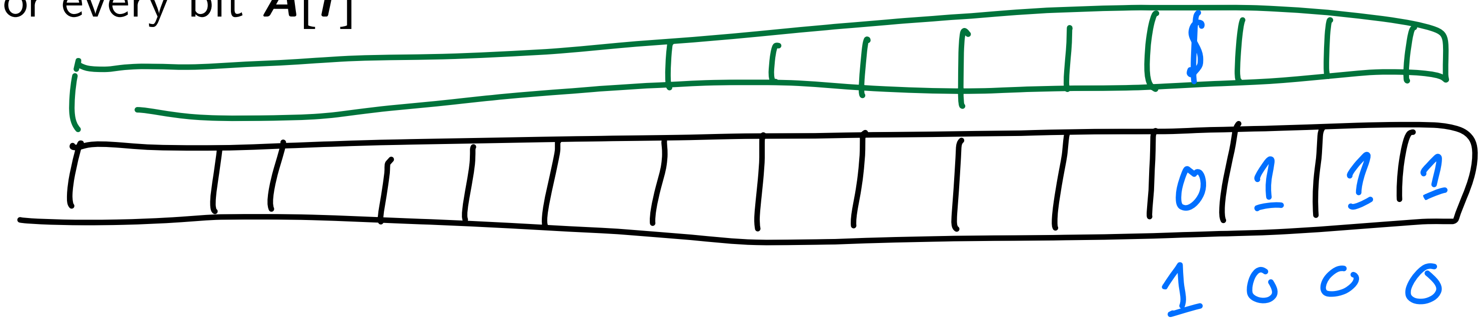
What about amortized cost?

Banks

Bank for every bit $\boldsymbol{A}[\boldsymbol{i}]$



Flip bit $\boldsymbol{i}$ from $\mathbf{0}$ to $\mathbf{1}$: add \$ to bank for $\boldsymbol{i}$
Flip bit $\boldsymbol{i}$ from $\mathbf{1}$ to $\mathbf{0}$: remove \$ from bank for $\boldsymbol{i}$

- No bank ever negative (induction)

Analysis

Do an increment, flips $\boldsymbol{k}$ bits $\Longrightarrow$ true cost is $\boldsymbol{k}$.

- # $\mathbf{0}$'s flipped to $\mathbf{1}$: 1
- # $\mathbf{1}$'s flipped to $\mathbf{0}$: $k-1$

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Flipping $\mathbf{1}$ to $\mathbf{0}$ paid for by bank! Costs $\mathbf{1}$, bank decreases by $\mathbf{1}$

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Global: Change in *total* bank is $-(\boldsymbol{k}-\mathbf{1})+\mathbf{1} = -\boldsymbol{k}+\mathbf{2}$
$\implies$ amortized cost $= \boldsymbol{c} + \boldsymbol{\Delta L} = \boldsymbol{k} + (-\boldsymbol{k}+\mathbf{2}) = \mathbf{2}$

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Do an increment, flips k bits $\implies$ true cost is k.

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$\implies$ amortized cost $= c + \Delta L = k + (-k + 2) = 2$

Potential function: let $\Phi = \#\mathbf{1}$'s in counter.
$\implies$ amortized cost $= c + \Delta\Phi = k + (-k + 2) = 2$

Example: Simple Dictionary

Setup

Same dictionary problem as last lecture (insert, lookup).

- Can we do something simple with just arrays (no trees)?
- Give up on worst-case: try for amortized.
  - Sorted array: inserts $\Omega(n)$ amortized (i'th insert could take time $\Omega(i)$)
  - Unsorted array: lookups $\Omega(n)$ amortized

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Solution: array of arrays!

- $A[i]$ either empty or a *sorted* array of *exactly* 2^i elements
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Example: insert $1 - 11$

$$A[0] = [5]$$
$$A[1] = [2, 8]$$
$$A[2] = \varnothing$$
$$A[3] = [1, 3, 4, 6, 7, 9, 10, 11]$$

Algorithm

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- While $A[i] \neq \emptyset$:
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  - $i++$
- Set $A[i] = B$

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Example: insert 12 into

$B = [2, 5, 8, 12]$

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- So after n inserts, have merged arrays of length 1 at most n times, arrays of length 2 at most $n/2$ times, arrays of length 4 at most $n/4$ times, ...

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$$\sum_{i=1}^{\lfloor \log n \rfloor} \frac{n}{2^{i-1}} 2^{i+1} = \Theta(n \log n)$$

- Amortized cost at most $\Theta(\log n)$!

Multiple Operations

How do we define amortized analysis of data structures with multiple operations?

Definition

If structure supports k operations, say that operation i has amortized cost at most α_i if for every sequence which performs with at most m_i operations of type i, the total cost is at most $\sum_{i=1}^{k} \alpha_i m_i$.

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- When analyzing multiple operations, need to use the same bank/potential for all of them!
- With multiple operations, bounds not necessarily unique. Different amortization schemes could yield different bounds, all of which are correct and non-contradictory.