

Lecture 7: Amortized Analysis

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601.433/633 Introduction to Algorithms

Introduction

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Data structures: *sequence* of operations!

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Last time: analyzed the (worst-case) cost of each operation.

What about (worst-case) cost of *sequence* of operations?

Definition & Example

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- ▶ Normal worst-case analysis: **100**
- ▶ Amortized cost: **$200/101 \approx 2$**

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- ▶ Normal worst-case analysis: **100**
- ▶ Amortized cost: **$200/101 \approx 2$**

If we care about total time (e.g., using data structure in larger algorithm) then worst-case too pessimistic

Amortized Algorithm

Still want worst-case, but worst-case over *sequences* rather than single operations.

Maybe only possible way to have an expensive operation is to have a bunch of cheap operations: amortized cost always small!

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Definition

If the amortized cost of every sequence of n operations is at most $f(n)$, then the *amortized cost* or *amortized complexity* of the algorithm is at most $f(n)$.

Example: Stack From Array

Stack Using Array

Stack:

- ▶ Last In First Out (LIFO)
- ▶ Push: add element to stack
- ▶ Pop: Remove the most recently added element.

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Building a stack with an array A :

- ▶ Initialize: $\text{top} = 0$
- ▶ Push(x): $A[\text{top}] = x$; $\text{top}++$
- ▶ Pop: $\text{top}--$; Return $A[\text{top}]$

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- ▶ Worst case: a single Push could cost $\Omega(n)$!

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New array has size $n + 1$:

- ▶ Sequence of n Push operations. Total cost: $\sum_{i=1}^n i = \frac{n(n+1)}{2} = \Theta(n^2)$.
- ▶ Amortized cost: $\Theta(n)$ (same as worst single operation!)

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Consider *any* sequence of n operations.

- ▶ Have to double when array has size $2, 4, 8, 16, 32, 64, \dots, \lceil \log n \rceil$
- ▶ *Total* time spent doubling: at most $\sum_{i=1}^{\lceil \log n \rceil} 2^i \leq 2n = \Theta(n)$
- ▶ Any operation that doesn't cause a doubling costs $O(1)$
- ▶ Total cost at most $O(n) + n \cdot O(1) = O(n)$
- ▶ Amortized cost at most $O(1)$

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Amortized analysis explains why it's better to double than add 1 !

More Complicated Analysis: Piggy Banks and Potentials

Basic Bank: Informal

Can be hard to give good bound directly on total cost.

- ▶ Lots of variance: some operations very expensive, some very cheap.
- ▶ Idea: “smooth out” the operations.
- ▶ “Pay more” for cheap operations, “pay less” for expensive ops.

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Charge cheap operations more, use extra to pay for expensive operations

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Bank L .

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So if $L_0 = \mathbf{0}$ and $L_n \geq \mathbf{0}$ (bank not negative): $\sum_{i=1}^n c_i \leq \sum_{i=1}^n c'_i$

- ▶ If $c'_i \leq f(n)$ for all i , then “true” amortized cost $(\sum_{i=1}^n c_i)/n$ also at most $f(n)$!

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Multiple banks

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Potential Functions:

- ▶ “Bank analogy”: we choose how much to deposit/withdraw.
- ▶ New analogy: “potential energy”. Function of state of system.
- ▶ Rename L to Φ : all previous analysis works same!
- ▶ Sometimes easier to think about: just define once at the beginning, instead of for each operation.

Example: Binary Counter

Binary Counter

Super simple setup: binary counter stored in array \mathbf{A} .

- ▶ Least significant bit in $\mathbf{A}[0]$, then $\mathbf{A}[1]$, ...
- ▶ Don't worry about length of array (infinite, or long enough)
- ▶ Only operation is increment.
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What about amortized cost?

Banks

Bank for every bit $A[i]$

Flip bit i from 0 to 1 : add $\$$ to bank for i

Flip bit i from 1 to 0 : remove $\$$ from bank for i

- ▶ No bank ever negative (induction)

Analysis

Do an increment, flips k bits \implies true cost is k .

- ▶ # **0**'s flipped to **1**:
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Global: Change in *total* bank is $-(k - 1) + 1 = -k + 2$

\implies amortized cost = $c + \Delta L = k + (-k + 2) = 2$

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Potential function: let $\Phi = \#1$'s in counter.

\implies amortized cost $= c + \Delta \Phi = k + (-k + 2) = 2$

Example: Simple Dictionary

Setup

Same dictionary problem as last lecture (insert, lookup).

- ▶ Can we do something simple with just arrays (no trees)?
- ▶ Give up on worst-case: try for amortized.
 - ▶ Sorted array: inserts $\Omega(n)$ amortized (i 'th insert could take time $\Omega(i)$)
 - ▶ Unsorted array: lookups $\Omega(n)$ amortized

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Solution: array of arrays!

- ▶ $A[i]$ either empty or a *sorted* array of *exactly* 2^i elements
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Example: insert **1 – 11**

$$A[0] = [5]$$

$$A[1] = [2, 8]$$

$$A[2] = \emptyset$$

$$A[3] = [1, 3, 4, 6, 7, 9, 10, 11]$$

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- ▶ Create array $B = [x]$
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- ▶ While $A[i] \neq \emptyset$:
 - ▶ Merge B and $A[i]$ to get B
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$$A[1] = \emptyset$$

$$A[2] = [2, 5, 8, 12]$$

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- ▶ So after n inserts, have merged arrays of length 1 at most n times, arrays of length 2 at most $n/2$ times, arrays of length 4 at most $n/4$ times, ...

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- ▶ Amortized cost at most $\Theta(\log n)$!

Multiple Operations

How do we define amortized analysis of data structures with multiple operations?

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If structure supports k operations, say that operation i has amortized cost at most α_i if for every sequence which performs with at most m_i operations of type i , the total cost is at most $\sum_{i=1}^k \alpha_i m_i$.

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- ▶ When analyzing multiple operations, need to use the same bank/potential for all of them!
- ▶ With multiple operations, bounds not necessarily unique. Different amortization schemes could yield different bounds, all of which are correct and non-contradictory.