Lecture 9: Disjoint Sets / Union-Find

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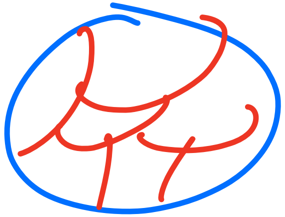
September 24, 2024
601.433/633 Introduction to Algorithms

Introduction



Informal: Universe of elements, want to maintain *disjoint sets*.

Slightly more formally:

- Make-Set($\boldsymbol{x}$): create a new set containing just $\boldsymbol{x}$ (i.e., $\{\boldsymbol{x}\}$)
- Union($\boldsymbol{x}, \boldsymbol{y}$): Replace set containing $\boldsymbol{x}$ ($\boldsymbol{S}$) and set containing $\boldsymbol{y}$ ($\boldsymbol{T}$) with single set $\boldsymbol{S} \cup \boldsymbol{T}$
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Rules: every set has a *unique* representative.

- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in same set, Find($\boldsymbol{x}$) = Find($\boldsymbol{y}$)
- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in different sets, then Find($\boldsymbol{x}$) $\neq$ Find($\boldsymbol{y}$)
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Note: disjoint (and partition) by construction!

Introduction (II)

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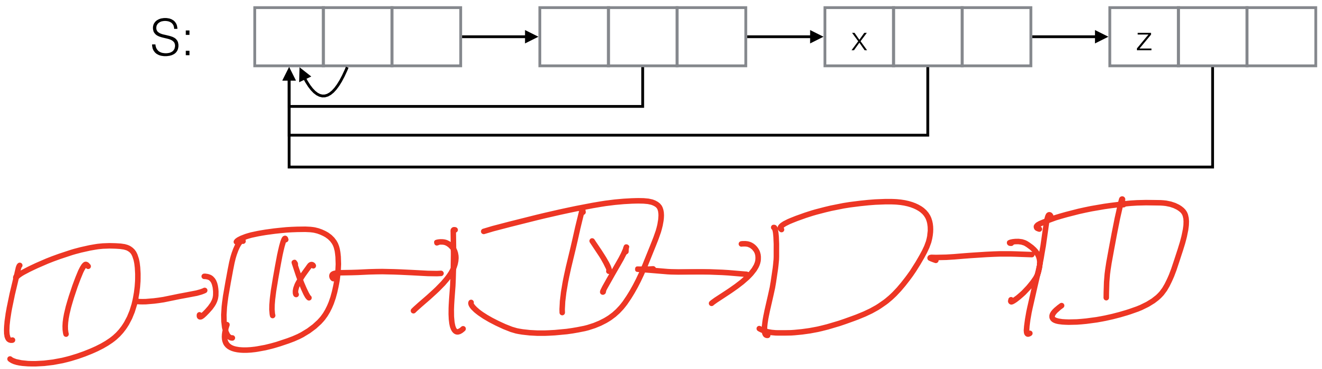
Notation and Notes:

- m operations total
- n of which are Make-Sets (so n elements)
- Assume have pointer/access to elements we care about (like last class)

First Approach: Lists

Linked list for each set.

- Representative of set is head (first element on list)
- Each element has pointer to head and to next element, so stored as triple: (element, head, next)



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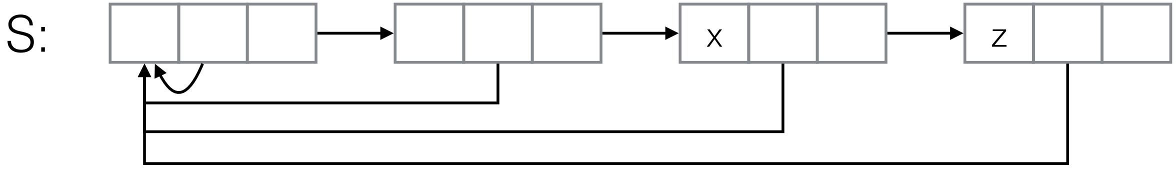
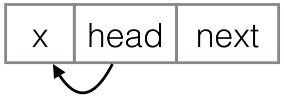


Make-Set(x):



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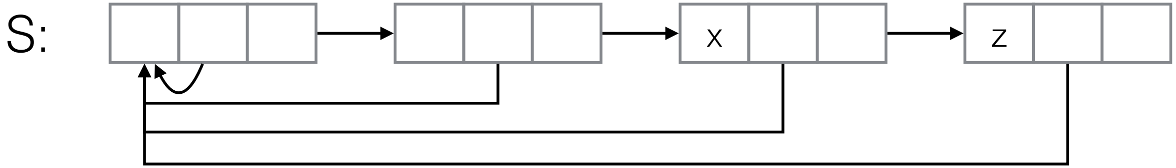

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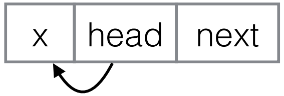
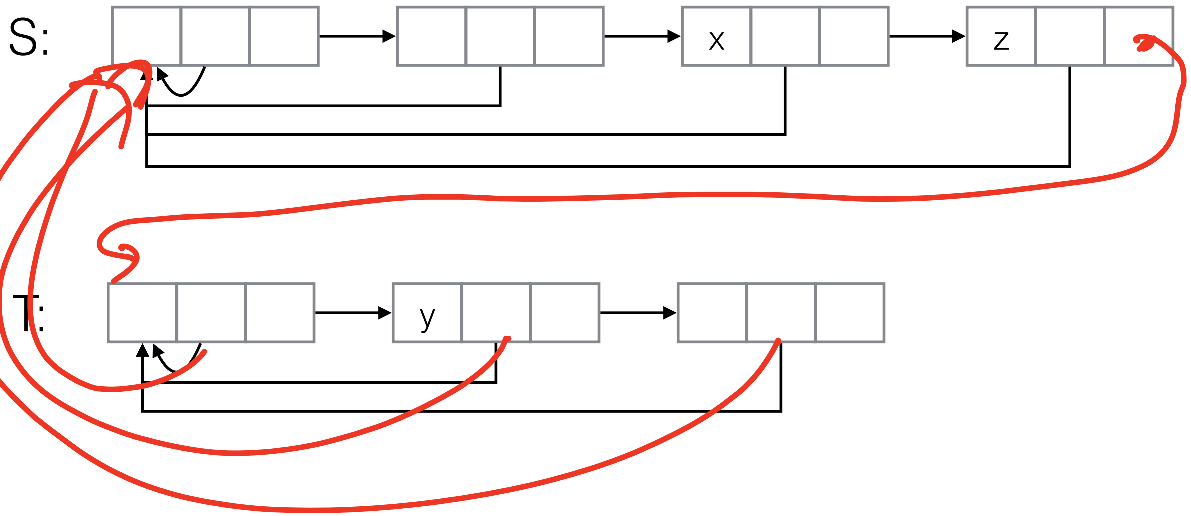
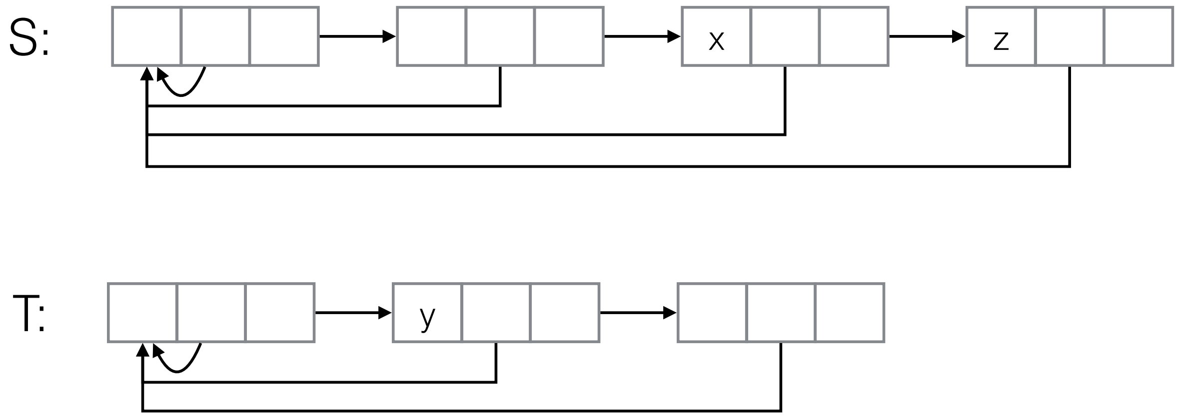


Find(x): return $x \rightarrow$ head

Union(*x*, *y*)



Union(x, y)



Obvious approach:

- Walk down S to final element z (starting from x)
- Set $z \rightarrow$ next = $y \rightarrow$ head
- Walk down T, set every elements head pointer to $x \rightarrow$ head

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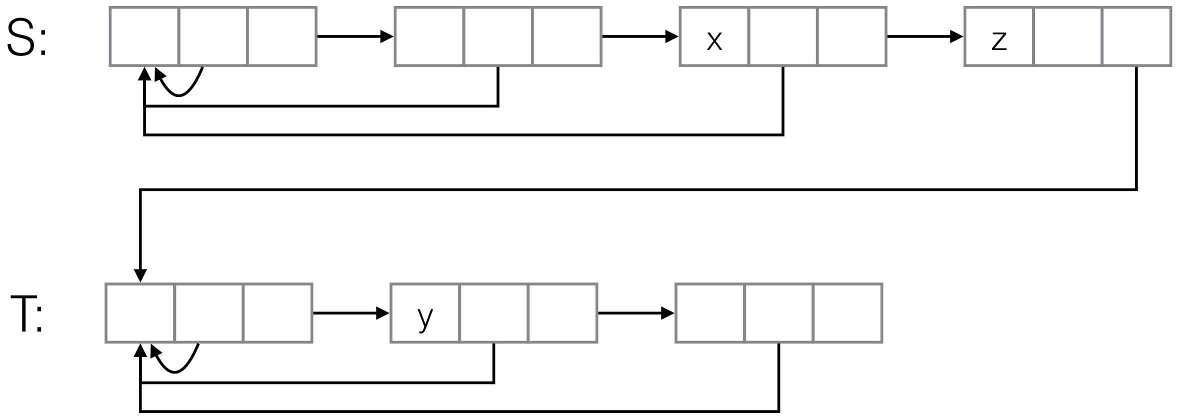

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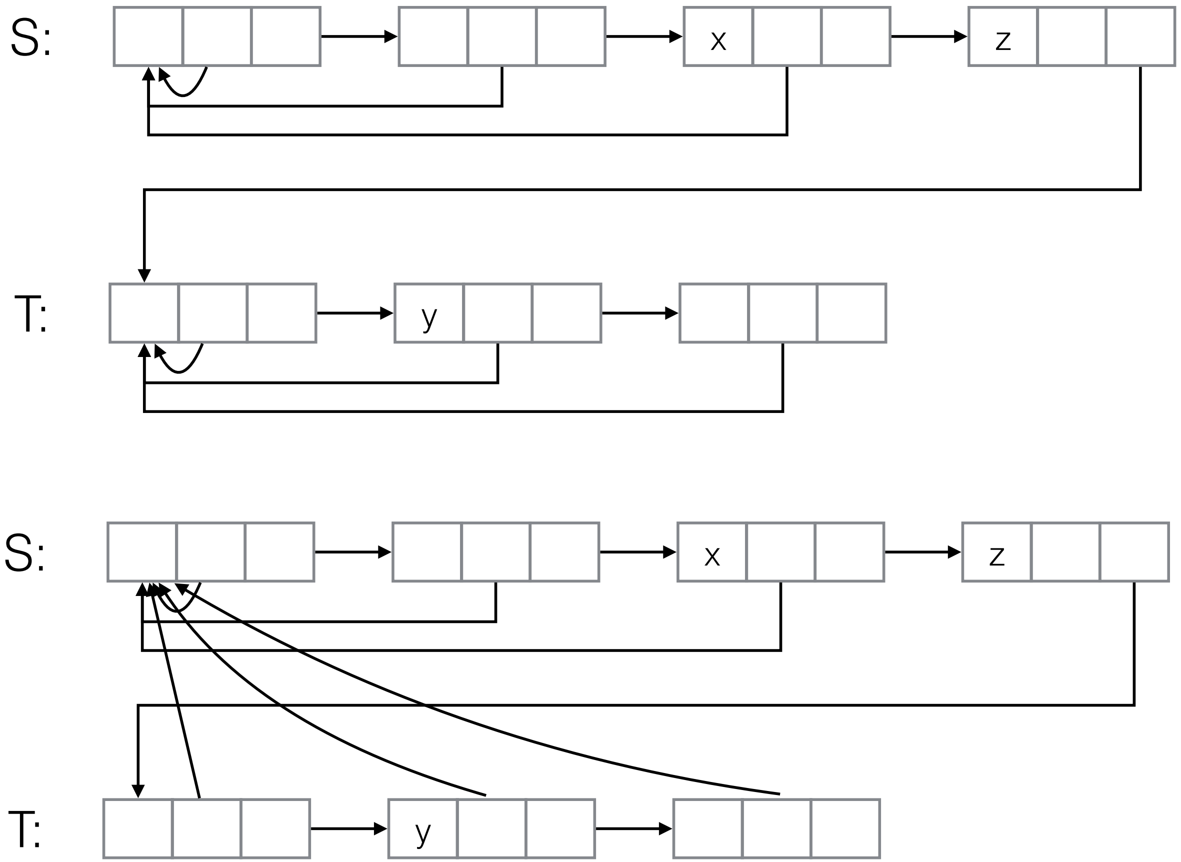

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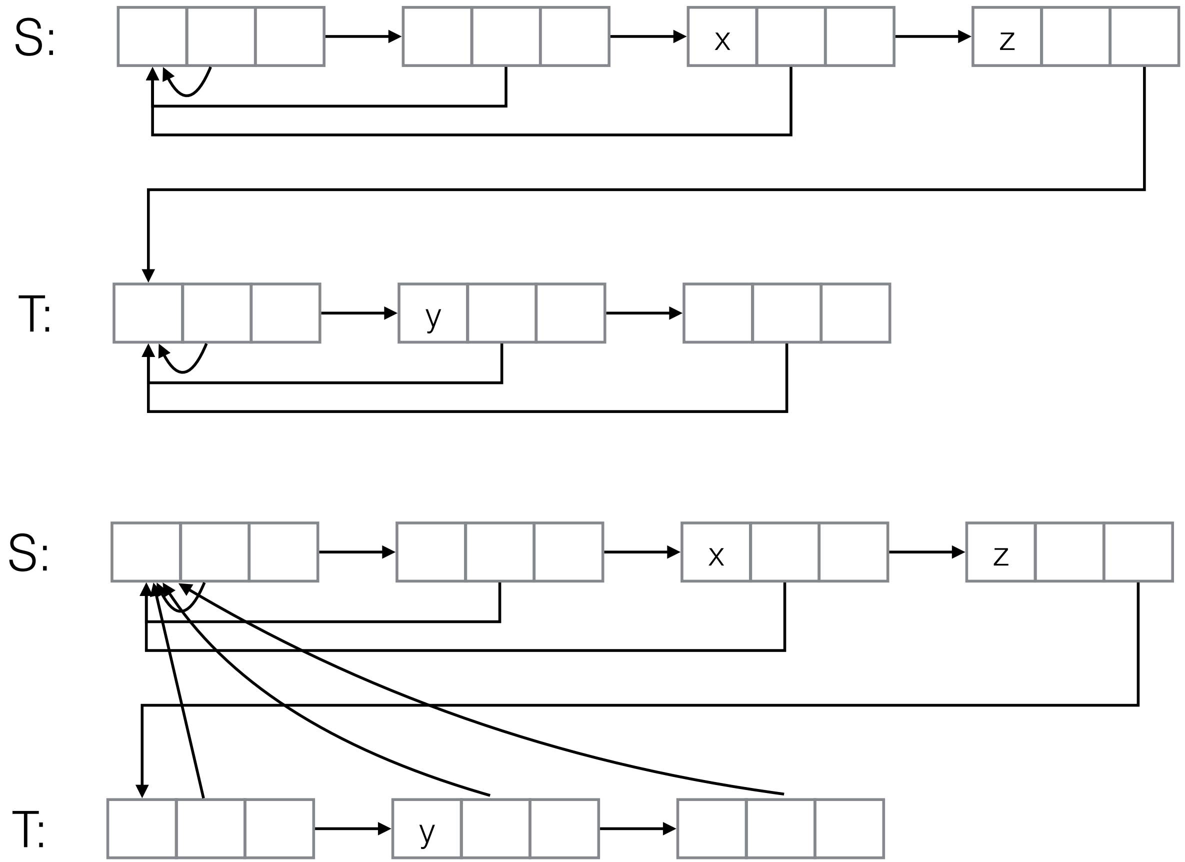
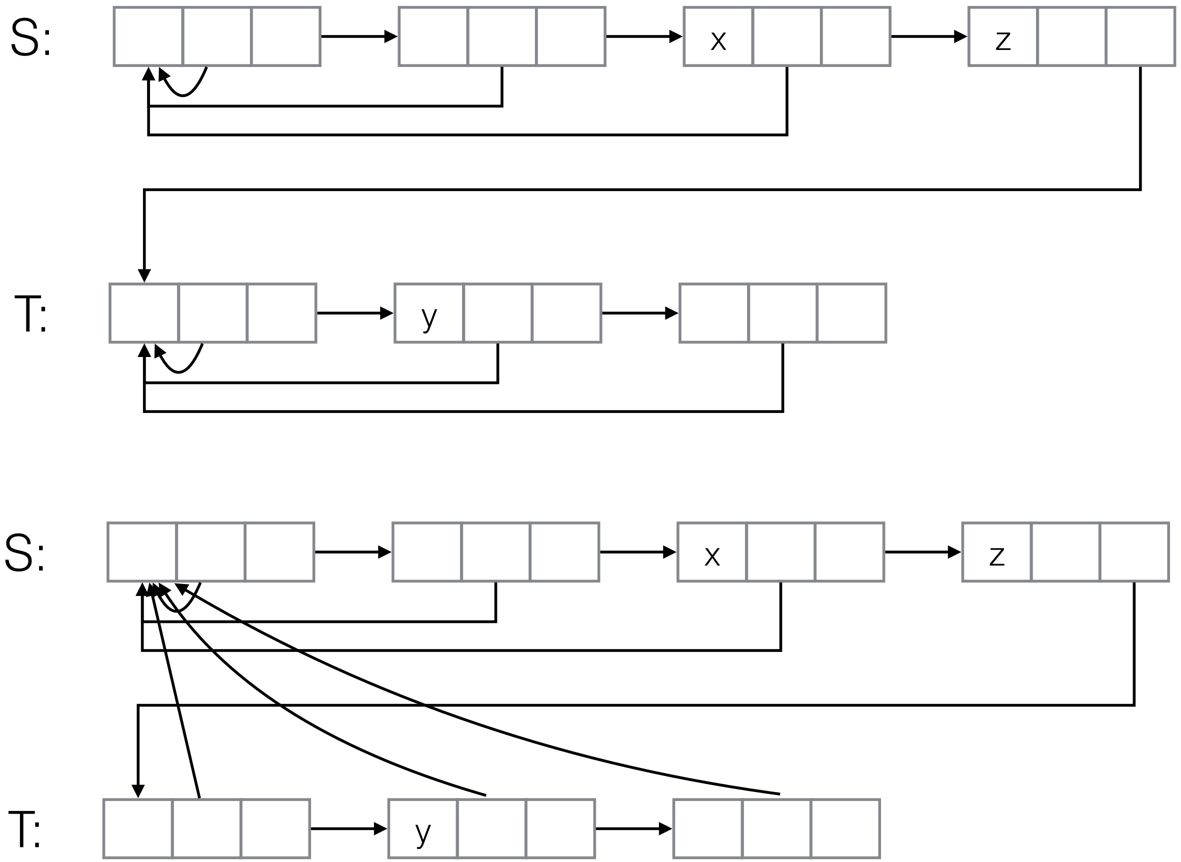


Running time:

Union($\boldsymbol{x}, \boldsymbol{y}$)



Running time: $\boldsymbol{O}(|\boldsymbol{S}| + |\boldsymbol{T}|)$

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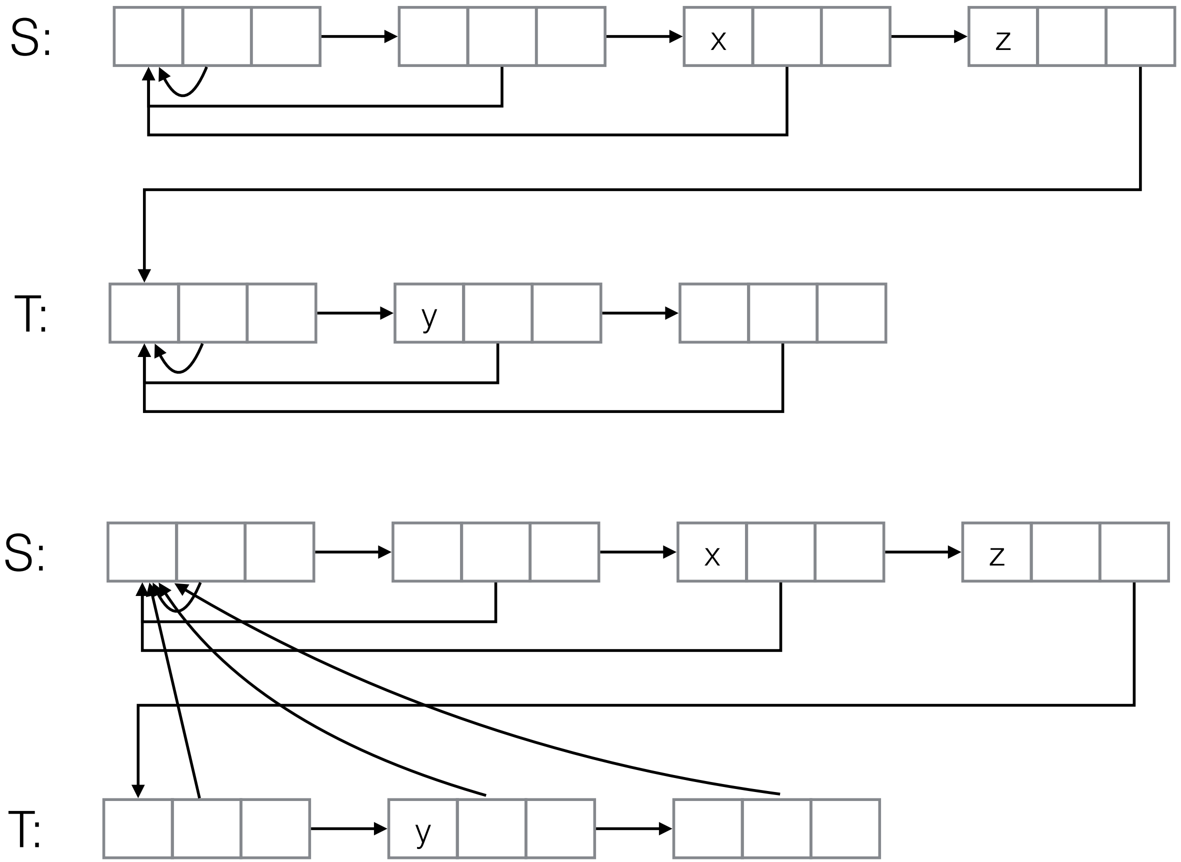

Running time: $O(|S| + |T|)$

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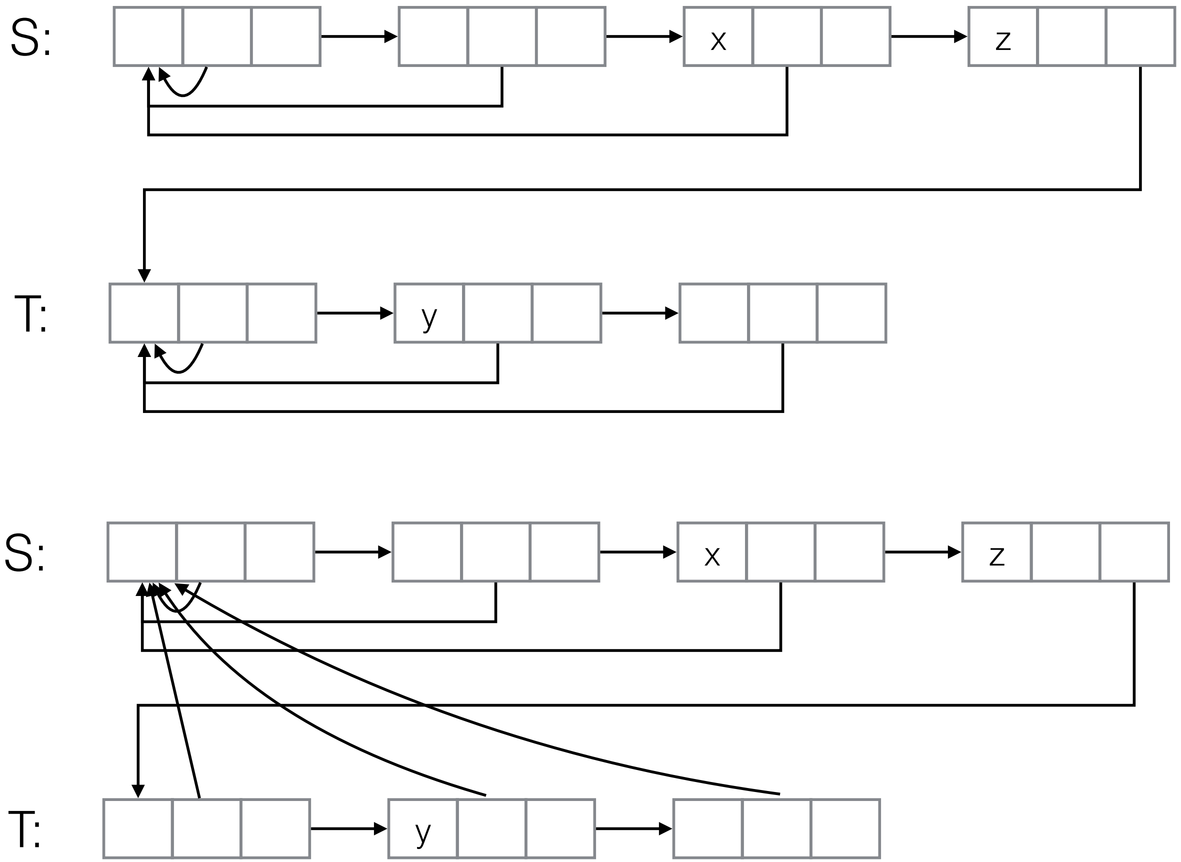

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Since $|S|, |T|$ could be $\Theta(n)$, can only say $O(n)$ for Unions

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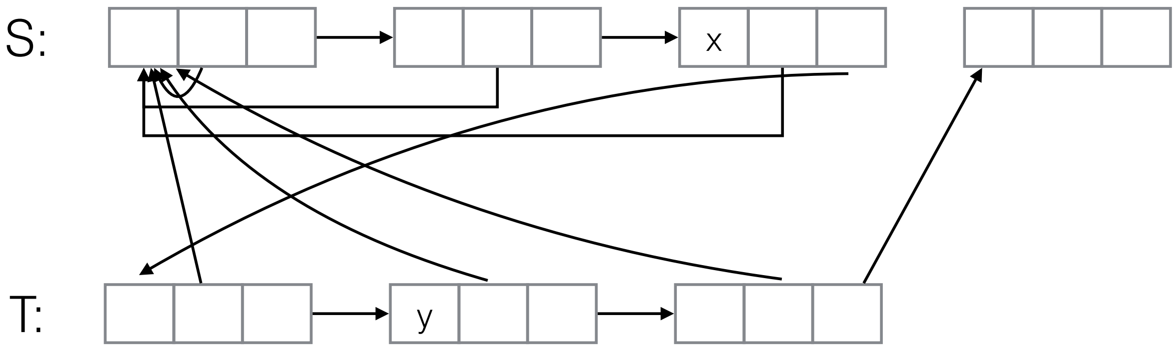

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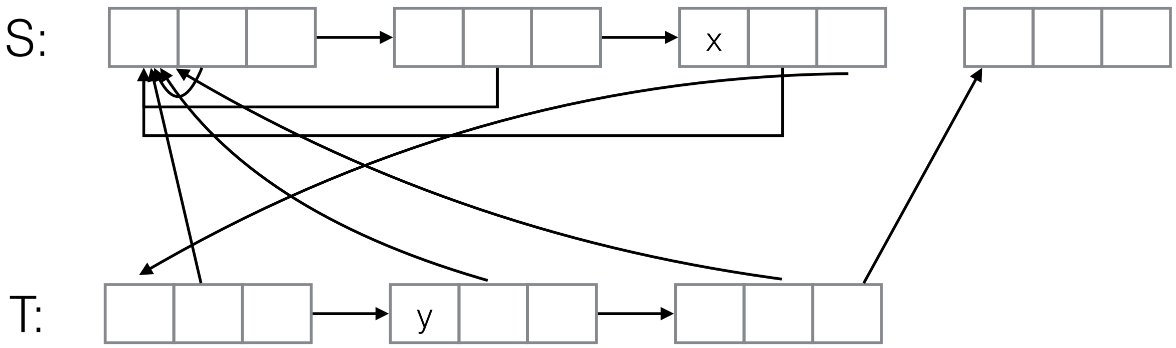

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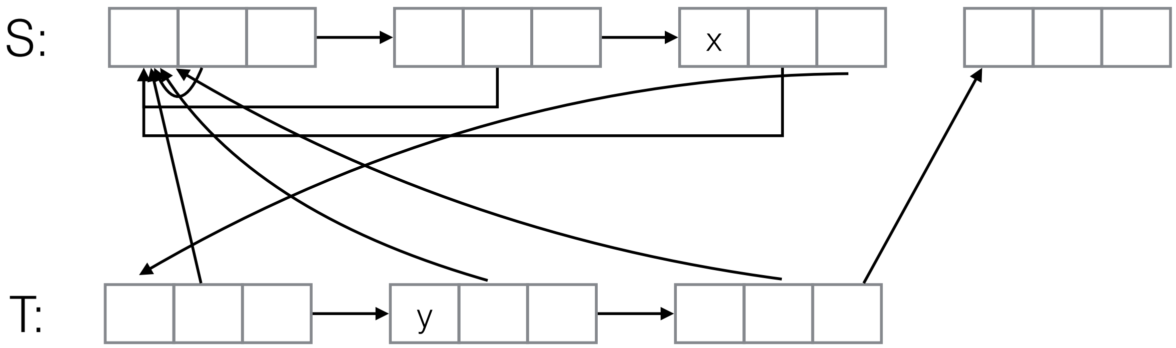

Running time: $O(|T|)$

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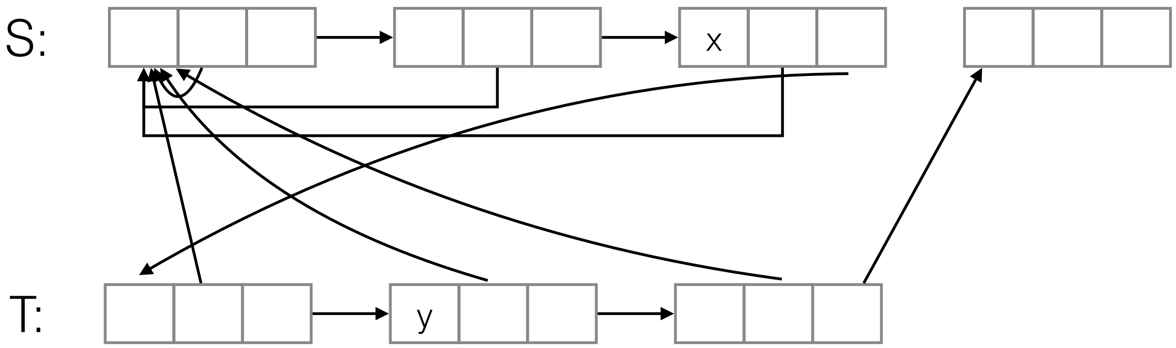

Running time: $O(|T|)$

- Still can't say anything better than $O(n)$

Even more improved Union(x, y)

Observation: Why splice T into S? Could also splice S into T.

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- Store size of set in head node.
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Theorem

The amortized cost of Find and Union is $O(1)$, and the amortized cost of Make-Set is $O(\log n)$.

Corollary

The total running time is $O(m + n \log n)$.

Amortized Analysis of List Algorithm

Banking/accounting argument: bank for every element

- When an element is created (via Make-Set), add $\log n$ tokens to its bank
- Find does not affect banks
- When doing Union, take token from bank of each element in smaller set.

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- Can only happen at most $\log n$ times. □

Amortized Analysis of List Algorithm (cont'd)

Make-Set:

- True cost: $O(1)$
- Change in banks: $\log n$

$\implies$ Amortized cost: $O(1) + O(\log n) = O(\log n)$

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Find:

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Union:

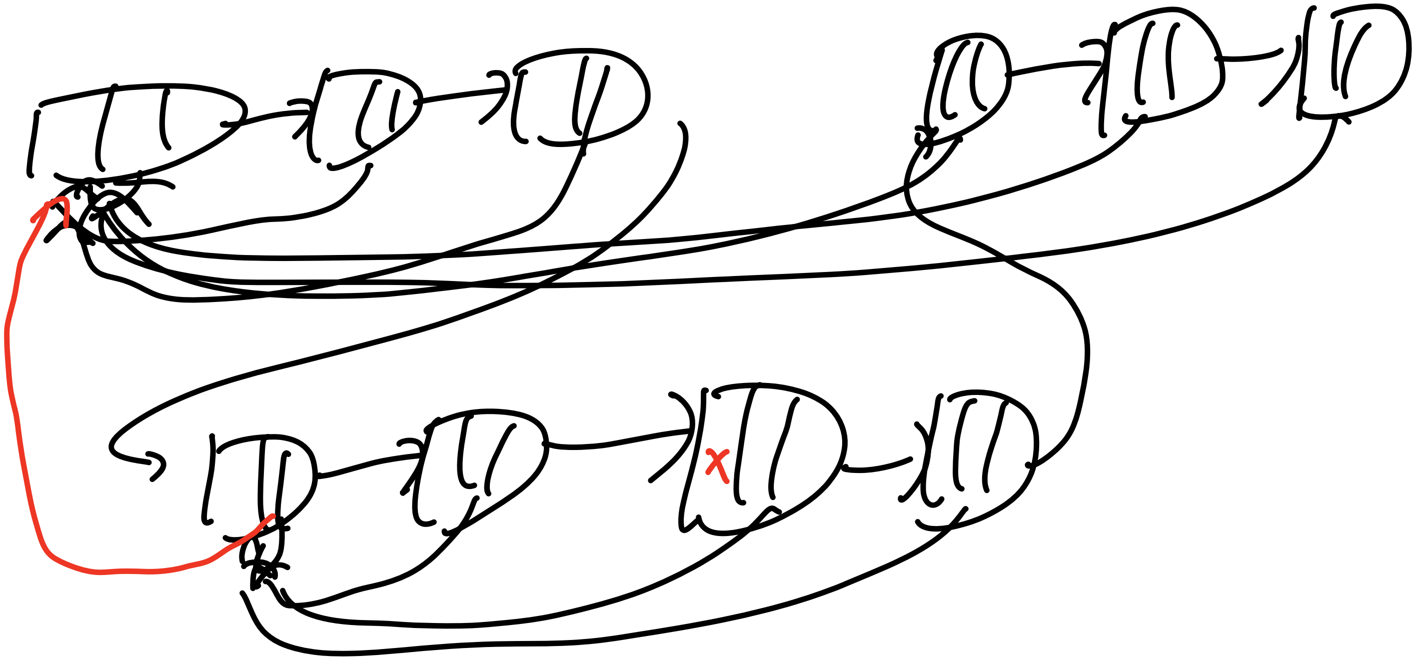
- True cost: $\min(|S|, |T|)$
- Change in banks: $-\min(|S|, |T|)$

$\Longrightarrow$ Amortized cost: $\min(|S|, |T|) - \min(|S|, |T|) = 0 = O(1)$.

Even Better

Starting idea: want to make Unions faster, willing to make Finds a little slower.

- Slow part of Union: updating all head pointers in smaller list.
- Don't do it!



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Idea 2: *Union By Rank*

- Size of set was important for lists, less important for trees.
- Choose which set to splice into which by *rank* of trees (related to height)

Main Result

Theorem

When using Path Compression and Union By Rank, total time at most $O(m \log^* n)$.

$\log^*$: iterated $\log_2$.

- $\log^* n = \#$ times apply $\log_2$ until get to 1

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Stronger theorem: total time at most $O(m \cdot \alpha(m, n))$.

- $\alpha(m, n)$: inverse Ackermann function. Grows even slower than $\log^*$.
- See CLRS for details

Formal Procedures: Make-Set and Find

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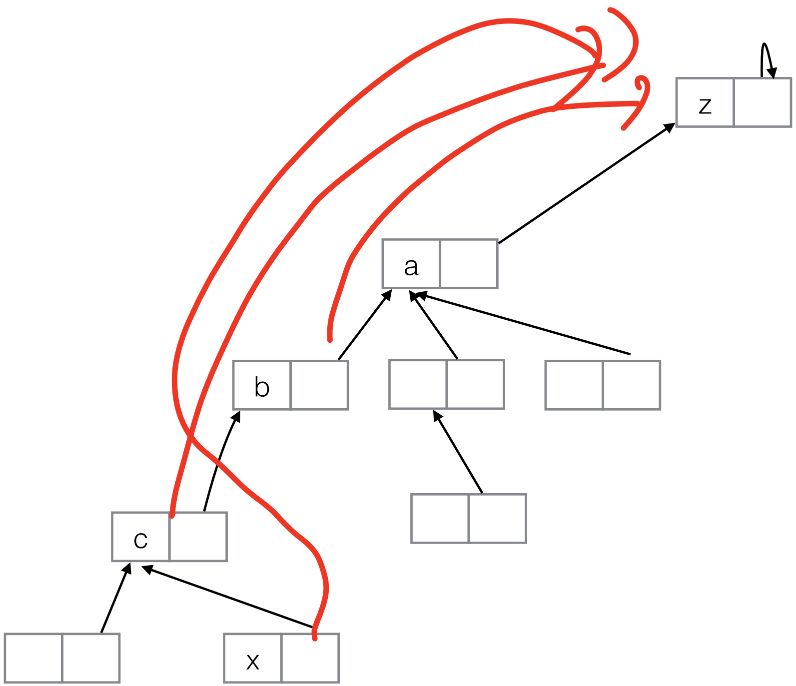
Running time of Find: depth of x (distance to root)

Find example



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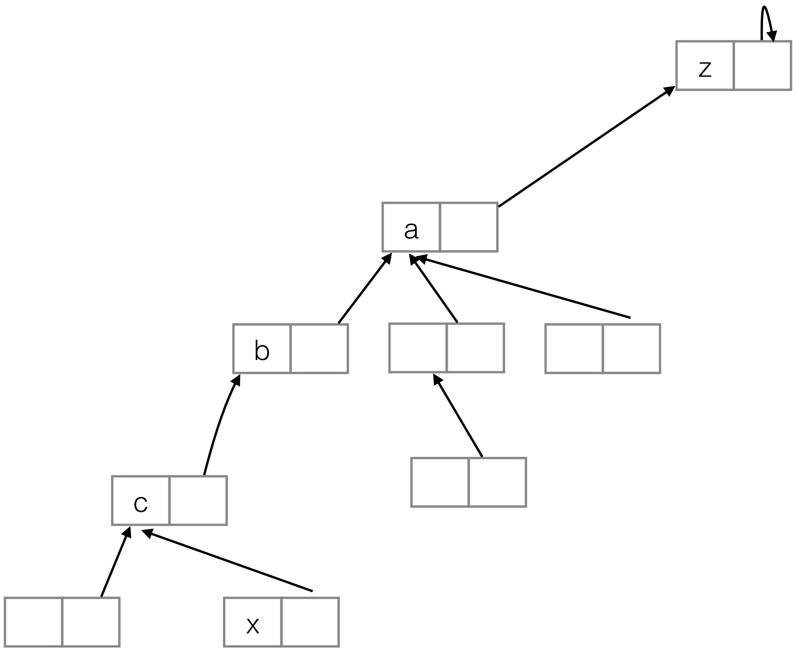

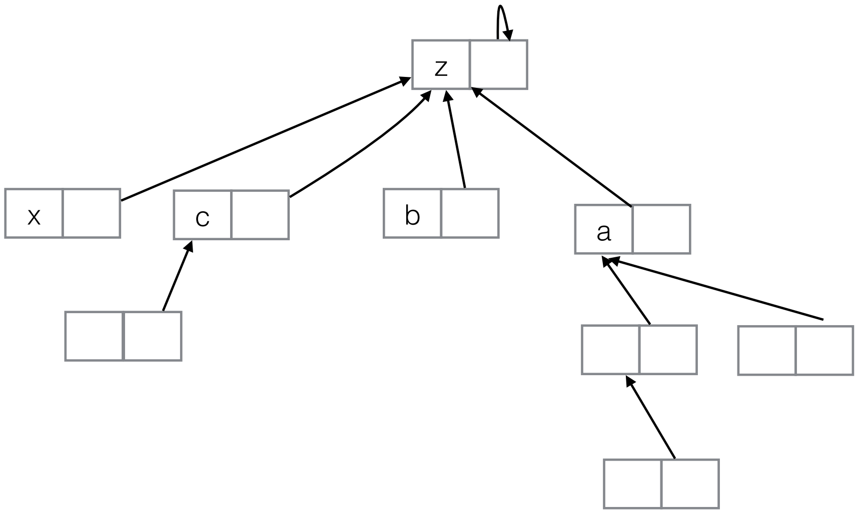

Formal Procedure: Union

Link($\boldsymbol{r_1, r_2}$): Only applied to root nodes

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Union($\boldsymbol{x}, \boldsymbol{y}$): Link(Find($\boldsymbol{x}$), Find($\boldsymbol{y}$))

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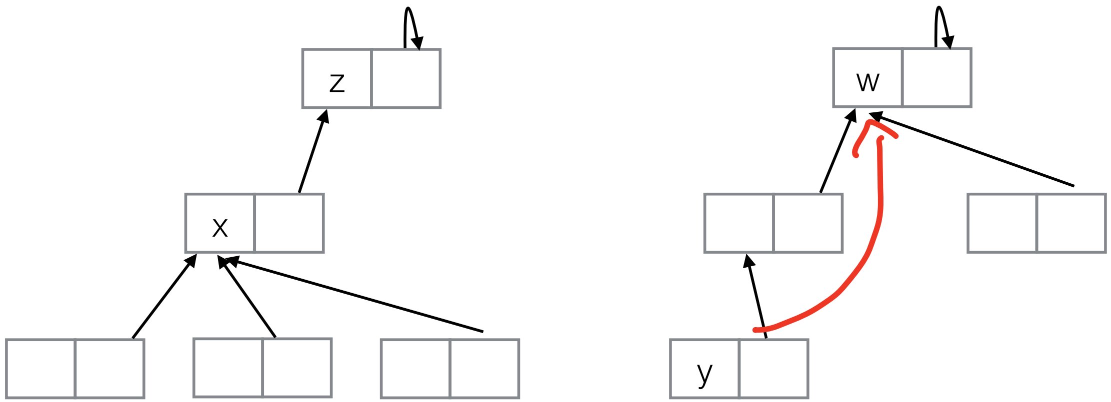
Union(x, y): Link(Find(x), Find(y))

- Running time: depth(x) + depth(y)

Union example



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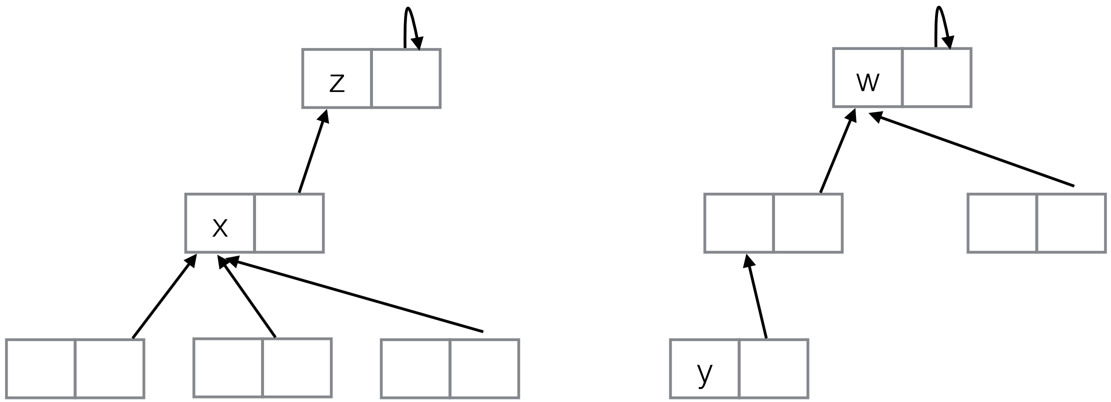

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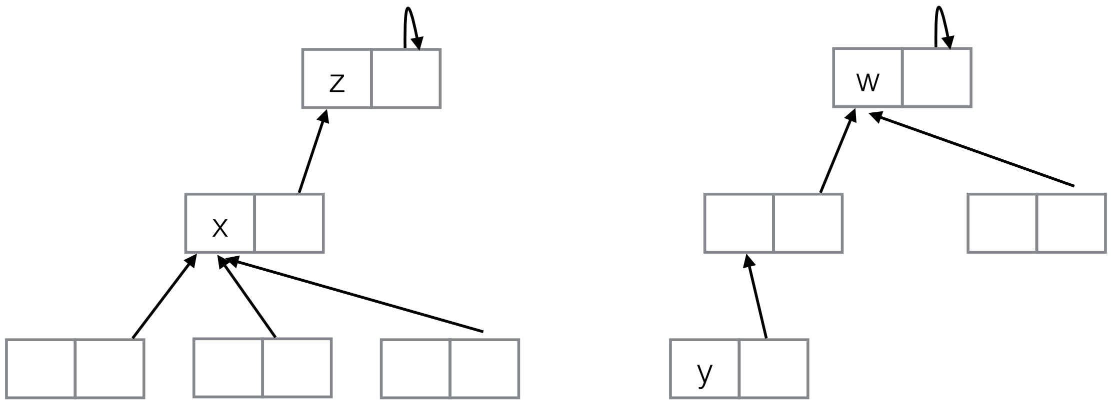

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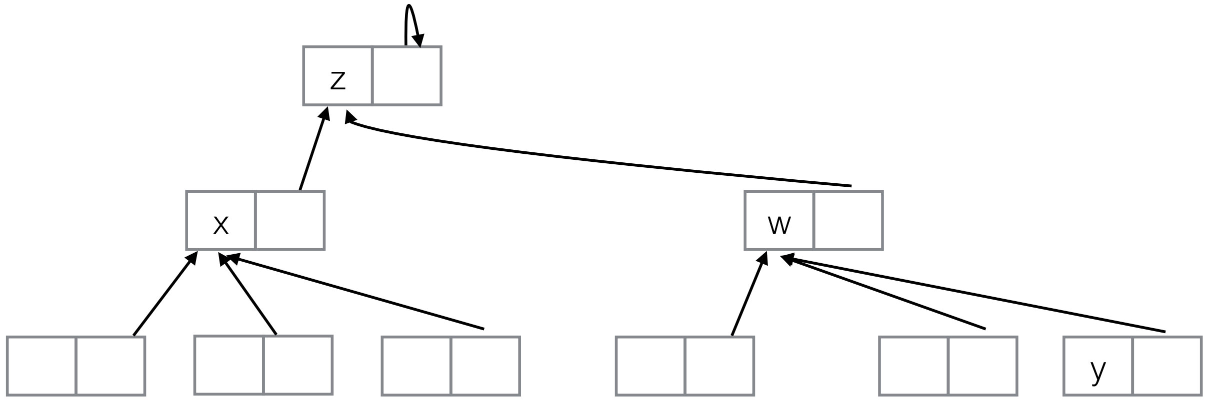

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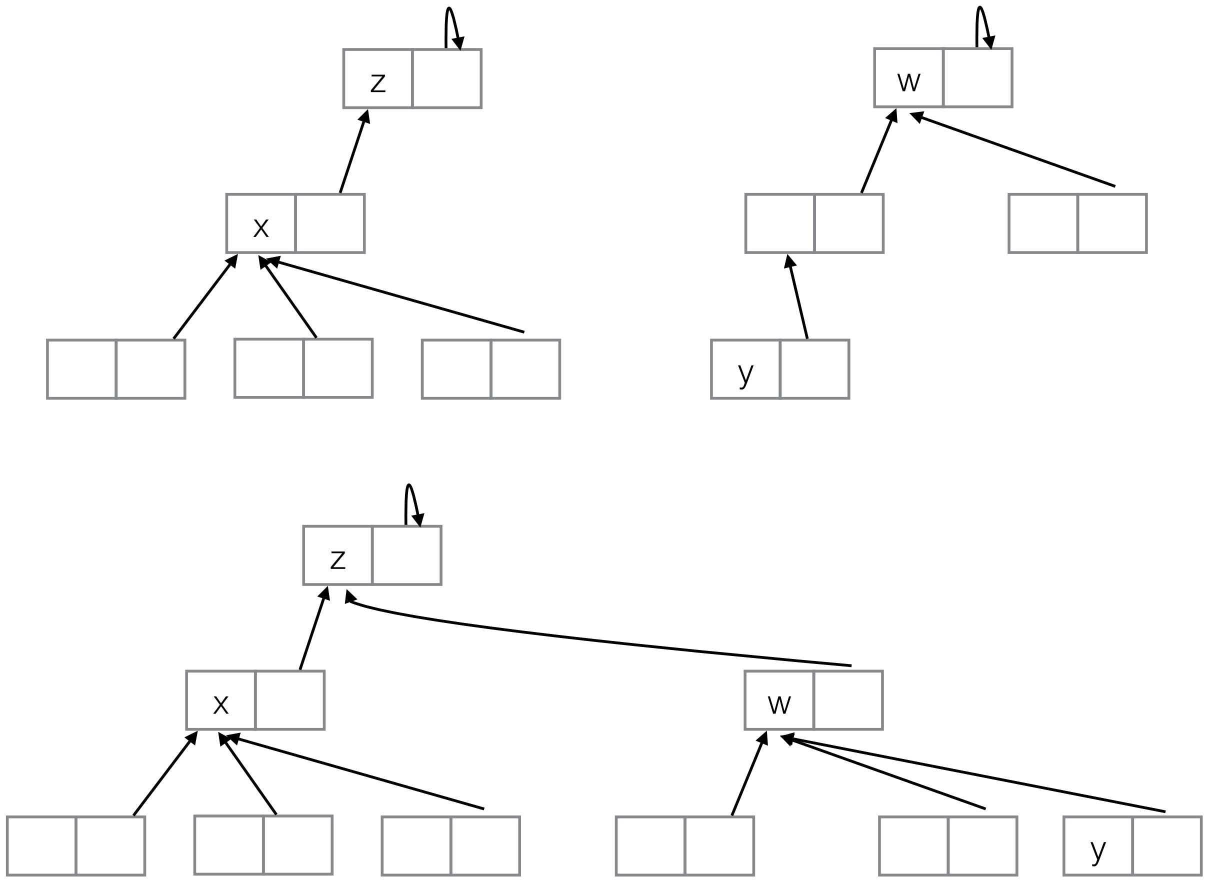

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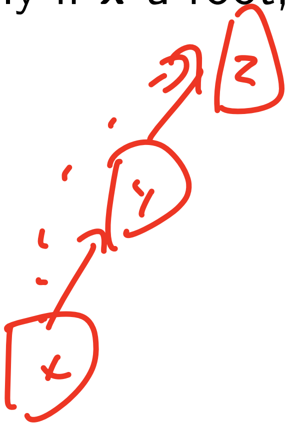
If $z \rightarrow \textbf{\textit{rank}} = w \rightarrow \textbf{\textit{rank}}$,
then $(z \rightarrow \textbf{\textit{rank}}){++}$

Properties of Ranks

1. If x not a root, then $(x \to rank) < (x \to parent \to rank)$
2. When doing path compression, if parent of x changes, new parent has rank strictly larger than old parent
3. $x \to rank$ can change only if x a root, and once x is a non-root it never becomes a root again.



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$\implies$ At least $2^{r-1} + 2^{r-1} = 2^r$ nodes in combined tree. □

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Lemma

There are at most $n/2^r$ nodes of rank at least r.

Proof.

Let x node of rank at least r. Let S_x be descendants of x when it first got rank r.
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So at most $2m$ Finds, want to bound total # parent pointers followed.

- At most one parent pointer to root per Find $\implies$ at most $O(m)$ parent pointers to roots.
- So only need to worry about parent pointers to non-roots.

Main Result II: Buckets

Put elements in buckets according to rank (only in analysis).

Notation: $\mathbf{2} \uparrow \boldsymbol{i}$ denote a tower of $\boldsymbol{i}$ **2**'s

- $\mathbf{2} \uparrow \mathbf{1} = \mathbf{2}, \quad \mathbf{2} \uparrow \mathbf{2} = \mathbf{2^2} = \mathbf{4}, \quad \mathbf{2} \uparrow \mathbf{3} = \mathbf{2^{2^2}} = \mathbf{2^4} = \mathbf{16}, \quad \mathbf{2} \uparrow \mathbf{4} = \mathbf{2^{2^{2^2}}} = \mathbf{2^{16}} = \mathbf{65536}$
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From Lemma: at most $n/(2^{2 \uparrow (i-1)}) = n/(2 \uparrow i)$ elements in bucket i.

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$$\sum_x \alpha(x) = \sum_{i=0}^{O(\log^* n)} \sum_{x \in B(i)} \alpha(x) \leq \sum_{i=0}^{O(\log^* n)} \sum_{x \in B(i)} (2 \uparrow i) \leq \sum_{i=0}^{O(\log^* n)} \frac{n}{2 \uparrow i}(2 \uparrow i) = O(n \log^* n)$$

$$\leq O(m\log^* n)$$